SS-9: Conditional Derivation of Simplicial Alpha-Polytope Connectivity from CPP Lattice Geometry

600-Cell Standard Model Emergence Series

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Abstract

Paper type: Derivation paper. Establishes a conditional theorem closing C4 (the simplicial-polytope contact structure assumption used implicitly in SS-7) on the refined-C1 (multi-faceted alpha rigidity, SS-7 v1.3) foundation, conditional on three new paper-level hypotheses registered as candidate open problems for first-principles closure.

SS-7 derived a zero-parameter formula $B(N_\alpha) = N_\alpha B_\alpha + (3N_\alpha - 6)B_{\text{pair}}$ for strict-$N{=}Z$ alpha-chain nuclei at $N_\alpha \in \{3, \ldots, 14\}$, where the edge count $3N_\alpha - 6$ is the simplicial-polytope edge count on N_α vertices. SS-7 used this edge count as a paper-level structural hypothesis (assumption C4) and tested it empirically against AME 2020. The zero-parameter agreement to within $\pm 1.5\%$ across twelve nuclei (RMS 0.80%) supported but did not derive the simplicial-polytope structure from CPP primitives. SS-9 advances OPEN-SS-24 (registered in SS-7 v1.0) toward closure.

The main result is a conditional theorem: under refined-C1 facets (a) and (b) (SS-7 v1.3), C2 (contact-face relation, SS-7), C3 (K_3 collective mode at each contact, SS-7), and four new paper-level hypotheses — C5 (ground-state energy minimization), C6 (cluster surface-realization), C7 (contact-graph planarity), C8 (FvdW centroid-realizability) — together with rigid packing and 3D-non-degeneracy, the ground-state contact graph $G(\mathcal{C})$ at $N_\alpha \in \{4, 5, 6, 7, 8, 9, 10, 12\}$ is the 1-skeleton of a simplicial convex 3-polytope with $|E| = 3N_\alpha - 6$, every face of the polytope a triangle, and the polytope is realized geometrically as the unique Freudenthal-van der Waerden convex deltahedron at N_α with vertices at the alpha LO centroids and uniform edge length $R_{\alpha\alpha}$. The proof routes through three lemmas: Lemma A (pairwise triangular contact, from C1′ + C2), Lemma C (energy minimization picks maximum edges, from C3 + C5), and Lemma B′ (contact graph = 1-skeleton of convex 3-polytope, from Lemma A + Lemma C + C6 + C7 + C8 via Euler's formula and Steinitz's theorem). Refined-C1 facet (b) is a necessary precondition for C8's plausibility at $N_\alpha \geq 7$, where the FvdW deltahedron has degree-≥ 5 vertices that strict-C1 with each alpha presenting four outer faces cannot host.

The conditional theorem promotes C4 from a B-tier structural hypothesis in SS-7's classification to a derived statement at the C5 + C6 + C7 + C8 + C1′ + C2 + C3 inheritance tier. Unconditional promotion is pending closure of OPEN-SS-29 (C5 from CPP), OPEN-SS-30 (C6 from CPP), OPEN-SS-33 (C7 from CPP), and the newly registered OPEN-SS-37 (C8 from CPP). The deltahedra-gap range $N_\alpha \in \{11, 13, 14\}$ is registered as OPEN-SS-31; clauses (i)–(iii) of the Theorem still apply (the contact graph remains a planar 3-connected simplicial graph), but clause (iv) does not (no FvdW deltahedron exists at those vertex counts). The $N_\alpha = 3$ planar degenerate case is handled separately. Refined-C1 facet (c) (cluster-level collective oblate-deformation slip-plane mode, OPEN-SS-32) is treated as an NLO addition to the LO binding formula and does not affect the LO geometric proof structure of this paper. Sessions 13–23 of the parallel OPEN-SS-32 / U-shape investigation thread are summarized in §8.3; standing best refinement at the time of v0.1 ship is Phase 8 Refinement A (factor 3.6 polytope-residual improvement, 48% of empirical scale captured).

Keywords: alpha-cluster model, simplicial polytope, Euler's formula, Steinitz's theorem, Freudenthal-van der Waerden deltahedra, contact graph, planarity, 3-vertex-connectivity, refined alpha rigidity, K_3 collective mode, conditional derivation, OPEN-SS-24, OPEN-SS-33, OPEN-SS-37, Conscious Point Physics, alpha-polytope edge formula.

Plain Language Summary: SS-7 found that the binding energies of medium-mass nuclei built from alpha particles match a clean formula: total binding equals the sum of individual alpha binding energies plus 2.342 MeV times the edge count of a closed polytope on N_α vertices, where each pair of touching alphas counts as one edge. The formula's edge count, $3N_\alpha - 6$, is the edge count of any triangulated sphere on N_α vertices (Euler's formula for simplicial convex 3-polytopes). SS-7 assumed without derivation that alpha clusters realize such polytopes and tested the assumption empirically; the test passed for twelve nuclei from carbon-12 through nickel-56 within ±1.5%. SS-9 asks: can the simplicial-polytope structure itself be derived from CPP primitives, rather than just matched empirically? The answer here is a conditional yes. Given (a) that alphas are leading-order rigid tetrahedral units with a multi-faceted rigidity that allows degree-≥ 5 vertex hosting (the SS-7 v1.3 refinement), (b) that contact between two alphas means face-to-face coincidence with one K_3 collective bond per contact (also from SS-7), (c) that the cluster minimizes total energy in its ground state, (d) that no alpha is interior to the cluster, and (e) that the resulting contact graph is planar (a topological consequence of the cluster being a closed shell of alphas), the simplicial-polytope structure follows by a chain of three lemmas. The energy-minimization condition forces the contact graph to have the maximum possible number of edges; Euler's formula then forces every face of the planar embedding to be a triangle; Steinitz's classical theorem (1922) then identifies the contact graph as the 1-skeleton of a convex 3-polytope; and the Freudenthal-van der Waerden classification (1947) identifies that polytope, at the eight vertex counts where a uniform-edge-length convex deltahedron exists, as the unique convex deltahedron with that vertex count. The five conditions become candidate open problems for first-principles derivation from the CPP axioms.

Contents

1 Introduction 4
1.1 Open Problems Addressed 4
1.2 Cascade context 4
1.3 What SS-9 delivers 5
1.4 What SS-9 does not deliver 5

2 Setup and Notation 6
2.1 Alpha-cluster configuration 6
2.2 Refined-C1 (multi-faceted alpha rigidity, inherited from SS-7 v1.3 §2.1) 6
2.3 Contact relation, contact graph, binding (inherited from SS-7) 7
2.4 Rigid packing 7
2.5 Paper-level structural hypotheses (the conditionals) 7
2.6 Sub-Lemma 2.1: Conditional derivation of C7 from cluster contractibility 9
2.7 Sub-Lemma 2.2: 3D-non-degeneracy via maximum-edge selection 11
2.8 Sub-Lemma 2.3: Well-definedness of the C5 ground state via compactness 13
2.9 Auxiliary assumptions 15

3 Lemma A: Pairwise Triangular Contact 15

4 Lemma C: Energy Minimization Picks Maximum Edges 16

5 Lemma B′: Contact Graph is 1-Skeleton of Convex 3-Polytope 16

6 Main Theorem (Conditional C4 Closure) 17

7 Scope Notes (Deltahedra-Gap, $N_\alpha = 3$ Degenerate, Coulomb) 20
7.1 The deltahedra-gap range $N_\alpha \in \{11, 13, 14\}$ 20
7.2 The $N_\alpha = 3$ planar degenerate case 20
7.3 Coulomb screening at the deltahedral surface 20

8 Honest Assessment of Closure Status 20
8.1 What v0.1 delivers 20
8.2 What v0.1 does not deliver 21
8.3 OPEN-SS-32 / U-shape investigation status (Phases 1–11, Sessions 13–23) 21
8.4 Net effect on programme scorecard 23

9 Gaps That Remain to Close Before SS-9 v1.0 23

10 Phase 4 Sketch: Programme-Level Closure Attempts 26

11 Physical Interpretation 27
11.1 CP/GP Signature at This Scale 28

12 CPP-to-Conventional-Physics Mapping 29

13 Conclusion 29
13.1 Swarm-Validation Contribution 30
13.2 Problem Status After This Paper 30

1 Introduction

Main result. Let $\mathcal{C}$ be the ground-state alpha-cluster configuration of $N_\alpha \in \{4, 5, 6, 7, 8, 9, 10, 12\}$ alphas, under refined-C1 facets (a) and (b), C2, C3, C5, C6, C7, C8, rigid packing, and 3D-non-degeneracy. Then:

(i) the contact graph $G(\mathcal{C})$ is the 1-skeleton of a simplicial convex 3-polytope;

(ii) $|E(\mathcal{C})| = 3N_\alpha - 6$;

(iii) every face of the polytope is a triangle;

(iv) the polytope is realized geometrically as the unique Freudenthal-van der Waerden convex deltahedron at N_α, with vertices at the alpha LO centroids and uniform edge length $R_{\alpha\alpha}$.

The proof routes through three lemmas (§§3–5) and the Freudenthal-van der Waerden classification of convex deltahedra. Refined-C1 facet (b) is load-bearing in clause (iv) at $N_\alpha \geq 7$.

1.1 Open Problems Addressed

This paper addresses OPEN-SS-24 (derivation of simplicial contact structure from CPP), originally registered in SS-7 v1.0 as a target for SS-9. The advancement is from "structural hypothesis to be verified empirically" (SS-7 status) to "conditional theorem at the C5 + C6 + C7 + C8 + C1′ + C2 + C3 inheritance tier" (SS-9 status), with four new candidate open problems registered for first-principles closure:

- OPEN-SS-29: derive C5 (ground-state energy minimization) from CPP axioms A1–A11.
- OPEN-SS-30: derive C6 (cluster surface-realization, no alpha interior to cluster) from A1–A11.
- OPEN-SS-33: derive C7 (contact-graph planarity) from A1–A11. *New this paper at v0.1; advanced to "conditionally closed modulo (H4) cluster contractibility and (H5) alpha-surface adjacency from A1–A11 + C5" at v0.2 via Sub-Lemma 2.1.*
- OPEN-SS-37: derive C8 (FvdW centroid-realizability) from A1–A11. *New at v0.5; surfaced by Session 28 ChatGPT review of v0.4 as the previously implicit Steinitz-to-centroid realization gap. Closure route involves showing the alpha cluster physics realizes the FvdW deltahedron geometry at the centroid positions; refined-C1 facet (b) is a necessary precondition at $N_\alpha \geq 7$.*

The deltahedra-gap range $N_\alpha \in \{11, 13, 14\}$, where clause (iv) of the Theorem does not apply because no FvdW convex deltahedron exists at those vertex counts, is registered as OPEN-SS-31. Refined-C1 facet (c) (cluster-level collective oblate-deformation slip-plane mode), an active investigation thread in parallel with this paper's development (OPEN-SS-32, SS-9 v0.3 §7 in earlier drafts; §8.3 of this paper), is treated as NLO and does not affect the LO geometric proof structure of this paper.

1.2 Cascade context

SS-5 [1] established that the light nuclei $A = 2, 3, 4$ form by face-to-face contact of hybrid-tetrahedral nucleons, with the cascade terminating at $A = 4$ because the regular tetrahedron is the unique closed 3-polytope on 4 vertices. SS-7 [2] extended this paradigm one level up: alpha particles themselves act as rigid tetrahedral building blocks, assembling into closed polytopes of alphas with K_3 base-to-base bonds across each shared face. The SS-7 binding formula

$$B(N_\alpha) = N_\alpha \cdot B_\alpha + (3N_\alpha - 6) \cdot B_{\text{pair}}, \tag{1}$$

with $B_\alpha = 28.296$ MeV (from SS-5) and $B_{\text{pair}} = M_0/\varphi = 2.342$ MeV (the SS-5 nucleon-pair binding quantum), reproduces twelve strict-$N{=}Z$ alpha-chain binding energies (^{12}C through ^{56}Ni) within ±1.5% at zero fitted parameters, with RMS error 0.80%.

The edge count $3N_\alpha - 6$ in (1) is the simplicial edge count on N_α vertices: a consequence of Euler's formula for convex 3-polytopes with triangular faces. The empirical agreement strongly supports the simplicial-polytope structure, but does not derive it from CPP primitives. SS-7 v1.0 explicitly registered OPEN-SS-24 ("derivation of simplicial contact structure from CPP lattice geometry") as a target for a future paper. SS-9 is that paper.

1.3 What SS-9 delivers

- A clean conditional theorem (§6) deriving clauses (i)–(iv) above from the lemma stack, with no remaining argumentative gaps in the proof of the lemma stack itself.
- A three-lemma proof structure: Lemma A (pairwise triangular contact, §3), Lemma C (energy minimization picks max edges, §4), Lemma B′ (contact graph = 1-skeleton of convex 3-polytope, §5).
- Honest delineation between what is proved *conditional on the hypothesis stack* and what remains as candidate open problems for first-principles closure (§§8–9).
- A new paper-level hypothesis C7 (contact-graph planarity) at the same tier as C5 and C6, with OPEN-SS-33 registered for first-principles closure.
- Integration of refined-C1 facet (b) (multi-faceted alpha rigidity from SS-7 v1.3) as a load-bearing component of the geometric-realization clause at $N_\alpha \geq 7$, where the FvdW deltahedron has degree-≥ 5 vertices.
- A scope-note treatment (§7) of the deltahedra-gap range $N_\alpha \in \{11, 13, 14\}$ and the $N_\alpha = 3$ planar degenerate case.
- An investigation status report (§8.3) summarizing the OPEN-SS-32 / U-shape thread (Sessions 13–23, Phases 1–11) that ran in parallel with SS-9 development, targeting refined-C1 facet (c). At the time of v0.1 ship, single-session R3-channel refinement candidates were declared exhausted; the standing best refinement is Phase 8 Refinement A.
- A Phase 4 sketch (§10) outlining the structure of programme-level closure attempts for C5, C6, and C7 from CPP axioms.

1.4 What SS-9 does not deliver

- First-principles derivation of C5, C6, or C7 from CPP axioms A1–A11. Each is registered as a candidate open problem (OPEN-SS-29, OPEN-SS-30, OPEN-SS-33).
- Resolution of the deltahedra-gap at $N_\alpha \in \{11, 13, 14\}$. Registered as OPEN-SS-31.
- Identification of the specific facet (b) accommodation mechanism at degree-≥ 5 vertices. Three candidate mechanisms (face-edge hybrid contact, K_3 delocalization across adjacent

faces, partial-overlap docking) are registered in SS-7 v1.3 §2.1; distinguishing them is testable via predicted contact-distance distributions at degree-5 sites, accessible to AMD or Brink–Bloch cluster-model calculations. Plausibly folds into the OPEN-SS-32 derivation if facets (b) and (c) share Layer-3 ancestry as the same K_3 scale-recurrence at different geometric venues.

- Closure of OPEN-SS-32 (refined-C1 facet (c) attenuation factor derivation). Active investigation thread, current status §8.3; standing best refinement Phase 8 Refinement A captures 48% of empirical polytope-residual scale; remaining 52% requires sub-shell-physics decomposition (multi-paper) or alternate-channel work outside R3 channel.

2 Setup and Notation

Symbols used throughout this paper:

- $N_\alpha \in \mathbb{N}$, $N_\alpha \geq 3$ — number of alpha clusters in the configuration
- L_α — alpha-internal scale (LO regular-tetrahedron edge length, from SS-5)
- $R_{\alpha\alpha} = 2.37$ fm — alpha-alpha contact distance (from SS-7, K_3 collective-mode minimization)
- $B_\alpha = 28.296$ MeV — ^{4}He single-alpha binding (from SS-5)
- $B_{\text{pair}} = M_0/\varphi = 2.342$ MeV — nucleon-pair binding quantum (from SS-5)
- $\mathcal{C} = (\alpha_1, \ldots, \alpha_{N_\alpha})$ — alpha-cluster configuration
- $G(\mathcal{C}) = (V, E)$ — contact graph
- $H(\mathcal{C}) = \text{conv}(c_1, \ldots, c_{N_\alpha})$ — convex hull of alpha LO centroids
- F_{ij} — shared LO triangular face at contact $\alpha_i \sim \alpha_j$

2.1 Alpha-cluster configuration

Let $N_\alpha \in \mathbb{N}$ with $N_\alpha \geq 3$. An N_α**-alpha cluster configuration** is a tuple $\mathcal{C} = (\alpha_1, \ldots, \alpha_{N_\alpha})$ of distinct rigid tetrahedral units in $\mathbb{R}^3$ ("alphas"), each at LO an approximately regular tetrahedron of edge length L_α (the SS-5-derived alpha-internal scale).

2.2 Refined-C1 (multi-faceted alpha rigidity, inherited from SS-7 v1.3 §2.1)

C1′ (refined alpha rigidity). Each α_i is a closed approximately regular 3-simplex with three structurally independent rigidity facets:

- **Facet (a)** — Internal LO rigidity. At LO the alpha is a regular tetrahedron with four equilateral triangular outer faces, with a $\sim 5\%$ residual band per the SS-5 LO-rigidity remark.
- **Facet (b)** — Vertex-hosting accommodation at degree-≥ 5 cluster vertices. When cluster topology requires the alpha to participate in ≥ 5 contacts, the additional contacts beyond four are hosted within the LO rigidity envelope via [mechanism TBD; candidates include face-edge hybrid contact, K_3 delocalization across adjacent faces, partial-overlap docking].

- **Facet (c)** — Cluster-level collective oblate-deformation slip-plane mode (provisional, OPEN-SS-32). Activates at belt/seam-supporting cluster shapes with binding contribution $\sim +B_{\text{pair}} \times$ attenuation factor.

For Lemma A, Lemma B′, Lemma C, and the Theorem below, the load-bearing content of C1′ is facet (a) plus facet (b). Facet (c) corrections enter as an NLO addition to the binding formula and are accounted for separately at OPEN-SS-32 closure tier; *facet (c) does not affect the LO geometric proof structure of this paper.*

2.3 Contact relation, contact graph, binding (inherited from SS-7)

C2 (base-to-base contact). The **contact relation** $\sim$ on $\{\alpha_1, \ldots, \alpha_{N_\alpha}\}$ is defined: $\alpha_i \sim \alpha_j$ iff there exist faces $F_i \subset \alpha_i$ and $F_j \subset \alpha_j$ such that $F_i \equiv F_j$ as triangular regions of $\mathbb{R}^3$. Under C1′ facet (a), each contact at LO has centroid-centroid separation $R_{\alpha\alpha}$ (the SS-7-calibrated alpha-alpha contact distance set by the K_3 collective-mode minimization).

Define:

- The **contact graph** $G(\mathcal{C}) = (V, E)$ with $V = \{\alpha_1, \ldots, \alpha_{N_\alpha}\}$ and $E = \{\{\alpha_i, \alpha_j\} : \alpha_i \sim \alpha_j\}$.
- The **convex hull of centroids** $H(\mathcal{C}) = \text{conv}(c_1, \ldots, c_{N_\alpha}) \subset \mathbb{R}^3$, where c_i is the LO centroid of α_i.
- The **binding energy at LO** $B(\mathcal{C}) = N_\alpha B_\alpha + |E| B_{\text{pair}}$, with $B_\alpha = 28.296$ MeV and $B_{\text{pair}} = M_0/\varphi = 2.342$ MeV inherited from SS-5/SS-7.

C3 (K_3 collective mode at each contact). Each contact $\alpha_i \sim \alpha_j$ contributes one collective bonding mode at energy B_{pair}.

2.4 Rigid packing

The phrase **rigid packing** is used throughout this paper as a structural condition on physically realizable cluster configurations. A configuration $\mathcal{C}$ satisfies rigid packing iff:

1. **No interpenetration.** For all $i \neq j$, $\text{int}(\alpha_i) \cap \text{int}(\alpha_j) = \emptyset$. The interiors of distinct alpha tetrahedra do not overlap.
2. **Contact only on faces, edges, or vertices.** If $\alpha_i \cap \alpha_j \neq \emptyset$, the intersection $\alpha_i \cap \alpha_j$ is a face, edge, or vertex of both α_i and α_j (or a subset thereof). No partial face overlaps, no point-on-interior contacts.
3. **No centroid coincidence.** For all $i \neq j$, $c_i \neq c_j$. Centroids of distinct alphas occupy distinct positions in $\mathbb{R}^3$; equivalently, two alphas cannot occupy the same location modulo orientation.

Rigid packing is a closed condition on configuration space (the open exclusion regions are the complements). It is invoked alongside C2 (which specifies the form of the contact relation when alphas do touch) and is logically independent of the C5/C6/C7/C8 paper-level structural hypotheses.

2.5 Paper-level structural hypotheses (the conditionals)

C5 (ground-state energy minimization). Among all N_α-alpha cluster configurations $\mathcal{C}$ that are physically realizable (no alpha-alpha interpenetration; cluster connected through G), the realized ground state minimizes total energy, equivalently maximizes $B(\mathcal{C})$. *Programme-level closure registered as OPEN-SS-29 candidate.*

C6 (cluster surface-realization). All centroids c_i lie on the boundary ∂H of the convex hull H. Equivalently: no alpha is interior to the cluster. *Programme-level closure registered as OPEN-SS-30 candidate.*

C7 (contact-graph planarity). The contact graph $G(\mathcal{C})$ is planar — it admits an embedding in the plane (equivalently, on S^2) without edge crossings. *Programme-level closure registered as OPEN-SS-33 candidate (NEW this paper).*

C8 (FvdW centroid-realizability). At $N_\alpha \in \{4, 5, 6, 7, 8, 9, 10, 12\}$, the abstract simplicial convex 3-polytope structure derived in Lemma 5.1 admits a geometric realization in $\mathbb{R}^3$ with vertices at the alpha LO centroids c_i and uniform edge length $R_{\alpha\alpha}$. Equivalently: there exists a physically realizable N_α-alpha cluster configuration whose contact graph is the 1-skeleton of the FvdW convex deltahedron at N_α, geometrically embedded at the alpha centroids. *Programme-level closure registered as OPEN-SS-37 candidate (NEW at v0.5).*

Physical motivation for C8. Steinitz's theorem (1922) classifies graphs that are 1-skeletons of convex 3-polytopes *abstractly*: any simple, planar, 3-vertex-connected graph G is the 1-skeleton of *some* convex 3-polytope P. The polytope P is determined up to abstract combinatorial structure, but its *geometric realization* in $\mathbb{R}^3$ — in particular, whether P can be realized with vertices at *specifically* the alpha LO centroid positions c_i (which are determined by the physics of rigid packing + C2 face-coincidences, not chosen abstractly) — requires a separate argument. C8 records this geometric realizability claim as a paper-level hypothesis. The Freudenthal–van der Waerden theorem (1947) supplies the abstract polytope side: at $N_\alpha \in \{4, 5, 6, 7, 8, 9, 10, 12\}$, the unique simplicial convex 3-polytope on N_α vertices with all-equilateral-triangular faces and uniform edge length is the FvdW deltahedron. C2 supplies the uniform-edge-length condition $R_{\alpha\alpha}$. The remaining ingredient — that the alpha cluster physics actually realizes this geometry — is C8. Refined-C1 facet (b) (vertex-hosting accommodation at degree-≥ 5 vertices) is a necessary precondition for C8's plausibility at $N_\alpha \geq 7$, since strict-C1 with each alpha presenting four outer faces cannot host the degree-5 vertices of the FvdW deltahedra at $N_\alpha \geq 7$; whether facet (b) is also *sufficient* to construct the realization (not just remove the strict-C1 obstruction) is part of OPEN-SS-37.

Important caveat (added v0.6 per Copilot review). C8 is a **strong geometric hypothesis** requiring future first-principles work. Specifically: C8 is *not* derivable from C1′ + C2 + rigid packing alone (Steinitz delivers an abstract polytope, not a centroid-realization at the c_i positions), C8 is *not* guaranteed by refined-C1 facet (b) (facet (b) removes the strict-C1 obstruction at degree-≥ 5 vertices but does not constructively produce the realization), and C8's first-principles closure (OPEN-SS-37) sits parallel in conceptual weight to OPEN-SS-29 (C5 derivation), OPEN-SS-30 (C6 derivation), and OPEN-SS-33 (C7 derivation). Empirical support for C8 is strong: SS-7 Table 1 confirms the FvdW edge-count formula at all eight specified N_α values within $\pm 1.5\%$ for twelve nuclei; this is consistent with C8 holding empirically but does not constitute a derivation. Sub-task (e) external review and OPEN-SS-37 closure attempts are the natural follow-up routes.

Physical motivation for C7. Under C6, the cluster is a "shell" of alphas with no interior member. Under C1′ + rigid packing, the cluster is a contractible connected 3D region (no internal voids in the standard nuclear physics interpretation of an alpha cluster). Each alpha contributes outer faces to the cluster's exterior 2-surface Σ, with $\Sigma \cong S^2$ topologically (boundary of a contractible 3D region). The contact graph admits a natural embedding on Σ via the alpha-dual construction (each alpha as a point on its own outer-face region, each contact as an arc through the shared interior face). This embedding makes G planar. The C7 hypothesis records this topological inheritance from C6 + cluster contractibility, abstracted from the geometric details of the embedding so that Steinitz's theorem can be invoked directly. C7's first-principles closure from CPP primitives is the OPEN-SS-33 question; physical motivation alone justifies it as a paper-level hypothesis at the C5/C6 tier. *To be explicit (added v0.6 per Copilot review): C7 is not derived in this paper. C7 is not guaranteed by rigid packing alone — a generic 3D rigid-packed cluster has a contact graph that need not be planar in the absence of C6's shell condition. C7 is a paper-level structural hypothesis pending OPEN-SS-33 closure (advanced at v0.2 to "conditionally closed modulo (H4) cluster contractibility and (H5) alpha-surface adjacency from A1–A11 + C5" via Sub-Lemma 2.1).*

C5, C6, C7, C8 are **paper-level structural hypotheses at the SS-7 inheritance tier**. None is derived from CPP axioms A1–A11 in this paper; each is registered as an OPEN-SS-2X / 3X candidate for follow-up programme-level closure.

2.6 Sub-Lemma 2.1: Conditional derivation of C7 from cluster contractibility

The motivation paragraph in §2.4 sketches a topological argument: contractibility of the cluster region $K = \bigcup_i T_i$ (closed union of LO tetrahedra) implies $\partial K \cong S^2$, and the alpha-dual construction gives a planar embedding of G on ∂K. We now formalize this as a sub-lemma, isolating two precisely-stated topological hypotheses. Throughout, T_i denotes the closed LO tetrahedron at α_i.

- **(H4) Cluster contractibility.** The cluster region $K = \bigcup_{i=1}^{N_\alpha} T_i$ is a contractible compact 3-manifold with piecewise-linear boundary.
- **(H5) Alpha-surface adjacency (ASA).** For every K_3-bonded pair $\{\alpha_i, \alpha_j\} \in E$, the shared LO triangular face F_{ij} has at least one boundary edge that lies on $\Sigma = \partial K$. Equivalently: at least one of the three edges of ∂F_{ij} is not shared with any third tetrahedron T_k.

Lemma 2.1 (Conditional derivation of C7)**.** *Suppose C1′, C2, C6, (H4), and (H5) hold. Then the contact graph $G(\mathcal{C})$ is planar, i.e., C7 holds.*

Proof. **Step 1: $\Sigma \cong S^2$.** By (H4), K is contractible, so $\chi(K) = 1$. The boundary-Euler formula for compact orientable 3-manifolds with boundary,

$$\chi(K) = \tfrac{1}{2}\,\chi(\partial K),$$

gives $\chi(\Sigma) = 2$. Since K is connected and embedded in $\mathbb{R}^3$ with the contractibility of (H4), $\Sigma = \partial K$ is a connected closed orientable 2-manifold. The classification of closed orientable surfaces gives $\Sigma \cong \Sigma_g$ for some genus $g \geq 0$, and $\chi(\Sigma) = 2$ forces $g = 0$, hence $\Sigma \cong S^2$.

Step 2: External-face decomposition $\Sigma = \bigcup_i F_i^{ext}$. For each alpha α_i, define the **external face set**

$$F_i^{ext} = \bigcup \{F : F \text{ is a 2-face of } T_i,\ F \subset \Sigma\}.$$

Under C1′ + rigid packing, the tetrahedra T_i meet only on lower-dimensional faces, so each 2-face of each T_i is either internal (shared with exactly one other T_j) or external (lying on Σ). The F_i^{ext}

form a closed cover of Σ with pairwise interior-disjoint regions:

$$\Sigma = \bigcup_{i=1}^{N_\alpha} F_i^{ext}, \qquad \mathrm{int}(F_i^{ext}) \cap \mathrm{int}(F_j^{ext}) = \emptyset \text{ for } i \neq j.$$

We claim $F_i^{ext} \neq \emptyset$ for every i. Suppose for contradiction that T_i has all four faces shared with neighbors T_j. Then every 2-face of T_i borders another tetrahedron, so a small open neighborhood of any point of ∂T_i lies entirely in K. Combined with $\mathrm{int}(T_i) \subset K$, this gives $T_i \subset \mathrm{int}(K)$; in particular $c_i \in \mathrm{int}(K)$. But $K \subseteq H(\mathcal{C})$, so $c_i \in \mathrm{int}(H(\mathcal{C}))$, contradicting C6. Hence at least one face of T_i lies on Σ, and $F_i^{ext} \neq \emptyset$.

Step 3: Alpha-dual embedding. For each alpha α_i, choose a basepoint $p_i \in \mathrm{int}(F_i^{ext})$ in the interior of one external face. For each contact $\{\alpha_i, \alpha_j\} \in E$, the shared face F_{ij} is internal (not on Σ). By (H5), at least one edge $e_{ij} \subset \partial F_{ij}$ lies on Σ. The edge e_{ij} is a boundary edge of T_i's external region F_i^{ext} (since one face of T_i adjacent to e_{ij} is external) and similarly of F_j^{ext}. Pick a generic interior point $q_{ij} \in e_{ij}$. Since F_i^{ext} is a connected polygonal region of Σ with p_i in its interior and q_{ij} on its boundary, there is a simple path on F_i^{ext} from p_i to q_{ij}. Similarly there is a simple path on F_j^{ext} from q_{ij} to p_j. Concatenating gives an arc $\gamma_{ij} \subset F_i^{ext} \cup F_j^{ext}$ on Σ from p_i to p_j.

Step 4: Generic non-crossing. For two contacts $\{\alpha_i, \alpha_j\}, \{\alpha_k, \alpha_l\} \in E$ with disjoint endpoint sets, the arcs γ_{ij}, γ_{kl} are supported in disjoint region unions because the F_m^{ext} are pairwise interior-disjoint and the connecting boundary edges e_{ij}, e_{kl} are also distinct (since they lie in distinct internal-face boundaries). Hence $\gamma_{ij} \cap \gamma_{kl} = \emptyset$. For two contacts $\{\alpha_i, \alpha_j\}, \{\alpha_i, \alpha_k\}$ sharing alpha α_i, the arcs share basepoint p_i and pass through F_i^{ext} on disjoint segments (from p_i to e_{ij} versus p_i to e_{ik}, with $e_{ij} \neq e_{ik}$). Since $\mathrm{int}(F_i^{ext})$ is a 2-dimensional region, by generic choice of the path interiors the arcs meet only at p_i. The basepoint coincidence at p_i is the vertex incidence required by the embedding.

The resulting collection $\{p_i\}_{i=1}^{N_\alpha} \cup \{\gamma_{ij}\}_{\{\alpha_i,\alpha_j\} \in E}$ embeds the topological realization of $G(\mathcal{C})$ in $\Sigma \cong S^2$ as a planar graph. Therefore $G(\mathcal{C})$ is planar, i.e., C7 holds. □

Remark 2.2 (Residual content of OPEN-SS-33)**.** *Sub-Lemma 2.1 reduces the closure of OPEN-SS-33 (deriving C7 from CPP primitives A1–A11) to deriving the two topological hypotheses (H4) cluster contractibility and (H5) alpha-surface adjacency from A1–A11 + C5. Both have viable closure paths:*

- ***(H4) Cluster contractibility from C5 (energy minimization).*** *Non-contractible clusters fall into two failure modes: (i) clusters with internal voids (where the cluster encloses a region of DP-sea at lower density than the surrounding sea); (ii) clusters with toroidal handles (genus $g \geq 1$ surface). Mode (i) is energetically disfavored because the enclosed void has an internal boundary contributing additional surface energy without compensating bulk binding, and the void's DP-sea state is at lower density than equilibrium DP-sea (a strain energy cost). Mode (ii) requires the contact graph to support handle topology, which by Euler $|E| - |V| + |F| = 2 - 2g$ would require $|E|$ to exceed the planar bound $3|V| - 6$ for a triangulated surface; under C5 such excess edges would compete energetically with the simply-connected ground state. The argument reduces to showing that any handle-bearing or void-bearing N_α-alpha cluster configuration has strictly lower binding $B(\mathcal{C})$ than at least one contractible configuration with the same N_α. This is a direct combinatorial-energetic exercise on the contact-graph realization classes for each N_α.*

- ***(H5) Alpha-surface adjacency from C5 + LO geometry.*** *(H5) fails iff there exists a contact pair $\{\alpha_i, \alpha_j\}$ whose shared face F_{ij} has all three boundary edges shared with additional tetrahedra. Each shared edge requires a third tetrahedron T_k to also contain that edge. This "three-around-an-edge" configuration is geometrically constrained: the three (or more) alpha tetrahedra meeting at a common edge must fit together with a total dihedral angle $\leq 2\pi$, and each contributes its own pair of K_3 contacts at adjacent faces. The combinatorial enumeration of edge-shared multi-alpha local configurations is small (at most a handful of cases at each N_α), and direct comparison of binding energies under C5 shows that face-shared K_3-bonded configurations dominate the ground state. (H5) is thus a generic-geometry consequence of C5 at LO, with the residual exclusion analysis tractable case-by-case.*

The sub-lemma + remark structure thereby advances OPEN-SS-33 from "raw open" (Session 24 ratification) to "conditionally closed modulo (H4) cluster contractibility and (H5) alpha-surface adjacency from A1–A11 + C5," a typical intermediate stage in programme-level closure work. Both residual sub-targets are smaller in scope than C7 itself: each reduces to a focused energetic/combinatorial argument under C5 rather than requiring full first-principles A1–A11 derivation.

2.7 Sub-Lemma 2.2: 3D-non-degeneracy via maximum-edge selection

The auxiliary assumption "3D-non-degeneracy" (§2, "Auxiliary assumptions" below) is stated as an unproved hypothesis in v0.1: *the alpha centroids $c_1, \dots, c_{N_\alpha}$ do not all lie in a single plane.* We now formalize a sub-lemma showing that 3D-non-degeneracy is a derived consequence of the maximum-edge selection principle (Lemma 4.1) under the existing inheritance hypotheses, at $N_\alpha \geq 4$.

Lemma 2.3 (3D-non-degeneracy from maximum-edge selection)**.** *Let $N_\alpha \in \{4, 5, 6, 7, 8, 9, 10, 12\}$. Suppose C1′ (facet (a) at LO), C2, C3, C5, and C8 (FvdW centroid-realizability) hold. Then no ground-state N_α-alpha cluster configuration $\mathcal{C}_{\text{ground}}$ has all centroids $c_1, \dots, c_{N_\alpha}$ lying in a single 2-plane.*

Proof. The strategy is to show that any coplanar centroid arrangement realizes strictly fewer K_3-bonded contacts than the corresponding 3D arrangement at the same N_α, and therefore is not a binding-energy maximum (equivalently, not a ground state under C5).

Step 1: Coplanar-centroid degree bound. Suppose all centroids c_i lie in a common plane $P \subset \mathbb{R}^3$. Fix any alpha α_i with centroid $c_i \in P$, and consider the four outward face-normals $\hat{n}_i^{(1)}, \hat{n}_i^{(2)}, \hat{n}_i^{(3)}, \hat{n}_i^{(4)}$ of α_i at LO. By C1′ facet (a), α_i is at LO a regular tetrahedron, so the four face-normals point from c_i toward the four face-centroids, forming a regular dual tetrahedron of unit-norm directions. Any three of these four directions span $\mathbb{R}^3$ (they are three of four vertices of a regular tetrahedron centered at c_i, a non-degenerate 3-simplex). In particular, no 2-plane through c_i can contain three or more of the four face-normals: the plane P through c_i contains *at most two* of $\hat{n}_i^{(1)}, \dots, \hat{n}_i^{(4)}$.

By C2 (face-to-face contact), $\alpha_i \sim \alpha_j$ requires the centroid difference $c_j - c_i$ to be parallel to one of α_i's outward face-normals (and at distance $R_{\alpha\alpha}$ along it). For $c_j - c_i$ to lie in P, the corresponding face-normal $\hat{n}_i^{(k)}$ must lie in P. Hence at most two of α_i's four faces can host a contact whose partner centroid lies in P. Therefore $\deg_G(c_i) \leq 2$ in any coplanar contact graph.

Step 2: Planar edge bound. Summing the degree bound over all alphas and using the handshake lemma,

$$2|E_{\mathrm{planar}}(\mathcal{C})| = \sum_{i=1}^{N_\alpha} \deg_G(c_i) \le 2N_\alpha,$$

so $|E_{\mathrm{planar}}(\mathcal{C})| \le N_\alpha$ for any cluster configuration with all centroids coplanar.

Step 3: 3D edge bound (existing realizations). At $N_\alpha \in \{4, 5, 6, 7, 8, 9, 10, 12\}$, the Freudenthal–van der Waerden convex deltahedra (cf. Theorem 6.1 clause (iv)) realize 3D centroid arrangements with $|E_{\mathrm{3D}}(\mathcal{C}_{\mathrm{FvdW}})| = 3N_\alpha - 6$. Concretely, $|E_{\mathrm{FvdW}}| = 6, 9, 12, 15, 18, 21, 24, 30$ at $N_\alpha = 4, 5, 6, 7, 8, 9, 10, 12$ respectively. By C8 (FvdW centroid-realizability, §2), each FvdW deltahedron at the specified N_α values is physically realized as an alpha cluster configuration with N_α alphas at its vertices, edge length $R_{\alpha\alpha}$ between adjacent centroids, and all face-coincidences accommodated within C1′ (facets (a), (b)). The existence of these 3D realizations is precisely the content of C8 and is not derivable from C1′ + C2 + C3 + C5 alone — this is why C8 appears in the lemma hypothesis list.

Step 4: Strict edge gain in 3D. Comparing Steps 2 and 3,

$$|E_{\mathrm{3D}}| - |E_{\mathrm{planar}}| \ge (3N_\alpha - 6) - N_\alpha = 2N_\alpha - 6 \ge 2 \quad \text{for } N_\alpha \ge 4.$$

By the SS-7 binding formula and C3, $B(\mathcal{C}) = N_\alpha B_\alpha + |E(\mathcal{C})| B_{\mathrm{pair}}$, so the binding-energy gain of the 3D arrangement over any coplanar arrangement is at least $(2N_\alpha - 6)B_{\mathrm{pair}} \ge 2B_{\mathrm{pair}} = 4.684$ MeV. By Lemma 4.1 (energy minimization picks maximum edges) and C5 (ground state minimizes energy), the ground-state $\mathcal{C}_{\mathrm{ground}}$ realizes the maximum $|E|$ among all physically realizable configurations at fixed N_α. Since any coplanar configuration has $|E| \le N_\alpha < 3N_\alpha - 6$ at $N_\alpha \ge 4$, no coplanar configuration is a ground state. Therefore the centroids of $\mathcal{C}_{\mathrm{ground}}$ do not all lie in a single plane. □

Remark 2.4 (Tightness of the planar bound and the $N_\alpha = 3$ exception)**.** *The bound $|E_{\mathrm{planar}}| \le N_\alpha$ is essentially tight: a closed ring of $N_\alpha \ge 3$ alphas with consecutive face-to-face contacts in a single plane realizes $|E| = N_\alpha$ when geometrically closing-up is permitted by the dihedral-angle constraints, and a linear chain realizes $|E| = N_\alpha - 1$. At $N_\alpha = 3$, the planar realization $|E| = 3$ (equilateral triangle of centroids) coincides with $3N_\alpha - 6 = 3$, so the planar and 3D bounds agree, and 3D-non-degeneracy fails: ^{12}C as the planar-triangle ground state is consistent with maximum-edge selection at $N_\alpha = 3$. This is why the Theorem (§6) excludes $N_\alpha = 3$ from its scope and the $N_\alpha = 3$ degenerate case is treated separately in §7: the proof above relies on strict $|E_{\mathrm{planar}}| < |E_{\mathrm{3D}}|$, which holds at $N_\alpha \ge 4$ but not at $N_\alpha = 3$. The threshold $N_\alpha \ge 4$ in Lemma 2.3 is therefore sharp.*

Remark 2.5 (Refined-C1 facet (b) does not weaken the planar bound)**.** *Refined-C1 facet (b) (vertex-hosting accommodation at degree-≥ 5 cluster vertices) might appear to relax the LO regularity assumed in Step 1 of the proof. It does not affect the planar bound for two reasons. First, facet (b) activates only at degree ≥ 5; the planar bound establishes $\deg_G(c_i) \le 2$, so facet (b) is not invoked in the planar configuration. Second, facet (b) operates within the LO rigidity envelope (a $\sim 5\%$ residual band per the SS-5 LO-rigidity remark): the four face-normal directions remain $O(5\%)$-deformed from their LO regular-tetrahedron positions, preserving the geometric fact that any three remain non-coplanar at LO + facet (b). The planar bound Step 1 uses only that no three face-normals are coplanar, which is robust to facet (b)'s small accommodation deformations.*

Remark 2.6 (Effect on §9 "3D-non-degeneracy" gap)**.** *The §9 entry on 3D-non-degeneracy (Gap 4 in v0.1) flags the auxiliary assumption as "Should ideally be derived from 'cluster is genuinely 3-dimensional at $N_\alpha \ge 4$' via the maximum-edge selection (planar arrangements have fewer edges* *than 3D arrangements at $N_\alpha \geq 4$, so C5 picks 3D). Worth verifying as a sub-lemma."* *Sub-Lemma 2.3 delivers exactly that derivation: 3D-non-degeneracy is now a theorem under the existing inheritance hypotheses C1′ + C2 + C3 + C5, not an independent auxiliary assumption. The Theorem statement (§6) and Lemma 5.1 statement may continue to list "3D-non-degeneracy" as a condition for clarity at v0.2, but it is now derivable rather than assumed.*

2.8 Sub-Lemma 2.3: Well-definedness of the C5 ground state via compactness

C5 (§2) asserts that "the realized ground state minimizes total energy, equivalently maximizes $B(\mathcal{C})$." This presupposes that such a maximum exists — i.e., that the supremum $\sup_{\mathcal{C}} B(\mathcal{C})$ over physically realizable configurations is attained. We now formalize a sub-lemma showing well-definedness of the C5 ground state under the standard topology on cluster configurations.

Lemma 2.7 (Well-definedness of C5 ground state)**.** *Let $N_\alpha \geq 2$. Define the configuration space*

$$\mathrm{Conf}(N_\alpha) \;=\; \big\{(c_1, R_1, \ldots, c_{N_\alpha}, R_{N_\alpha}) \in (\mathbb{R}^3 \times \mathrm{SO}(3))^{N_\alpha} \;:\; \textit{rigid packing; } G(\mathcal{C}) \textit{ connected}\big\}/\mathrm{SE}(3),$$

i.e., N_α-tuples of (centroid, orientation) pairs satisfying no alpha-alpha interpenetration and G-connectedness, modulo the rigid motion group $\mathrm{SE}(3) = \mathbb{R}^3 \rtimes \mathrm{SO}(3)$. Then $\sup_{\mathcal{C}\in\mathrm{Conf}(N_\alpha)} B(\mathcal{C})$ is attained at some $\mathcal{C}^* \in \mathrm{Conf}(N_\alpha)$.

Proof. **Step 1: G-connectedness gives diameter bound.** G being connected means any two centroids c_i, c_j are linked by a path in G of length at most $N_\alpha - 1$ (a connected graph on N_α vertices has diameter at most $N_\alpha - 1$). Each edge of this path corresponds to a face-to-face contact, contributing centroid-distance $R_{\alpha\alpha}$ via C2. By the triangle inequality, $\mathrm{diam}(\{c_i\}) \leq (N_\alpha - 1)R_{\alpha\alpha}$. Modulo SE(3) (translation + rotation), the centroids fit inside a closed ball $\overline{B(0, (N_\alpha - 1)R_{\alpha\alpha})} \subset \mathbb{R}^3$ after centering on $c_1 = 0$. (Note: $|E| \geq N_\alpha - 1$ is necessary but not sufficient for connectedness — a triangle plus an isolated vertex on $N_\alpha = 4$ has $|E| = 3 = N_\alpha - 1$ but is disconnected — so we use G connected directly, not the edge-count proxy.)

Step 2: Pre-compactness of $\mathrm{Conf}(N_\alpha)$. The reduced configuration space (after SE(3) quotient) embeds into

$$\overline{B(0, (N_\alpha - 1)R_{\alpha\alpha})}^{N_\alpha - 1} \times \mathrm{SO}(3)^{N_\alpha},$$

a product of a compact subset of $(\mathbb{R}^3)^{N_\alpha - 1}$ (after centering) and a compact group. The product is compact. Rigid packing (no alpha-alpha interpenetration) is a closed condition: the open exclusion region $\mathrm{int}(\alpha_i) \cap \mathrm{int}(\alpha_j) \neq \emptyset$ is open, so its complement is closed. The intersection of a closed set with the compact ambient space is compact. Hence $\mathrm{Conf}(N_\alpha)$ is contained in a compact set $\overline{\mathrm{Conf}}(N_\alpha)$ obtained by including limit configurations where G-connectedness may degenerate.

Step 3: Upper-semi-continuity of B. The contact relation $\alpha_i \sim \alpha_j$ at LO requires exact face-coincidence between a face of α_i and a face of α_j. For each ordered pair (i, j) and each pair of face indices $(a, b) \in \{1, 2, 3, 4\}^2$, the condition "face a of α_i coincides with face b of α_j" is a system of equality constraints on (c_i, R_i, c_j, R_j): parallel face normals (with opposite orientation), centroid-to-centroid distance $R_{\alpha\alpha}$, and aligned in-face vertex correspondence. Each such condition defines a closed subvariety $F_{ij}^{ab} \subset \overline{\mathrm{Conf}}(N_\alpha)$. The contact pair is realized iff $\mathcal{C} \in \bigcup_{a,b} F_{ij}^{ab}$, a finite union of closed subvarieties, hence closed. Denote this union F_{ij}. The pair indicator $\mathbf{1}_{F_{ij}}$ is upper-semi-continuous (USC) since F_{ij} is closed.[1]

[1] For F closed and $x_n \to x$: if $x \in F$, then $\mathbf{1}_F(x) = 1 \geq \limsup \mathbf{1}_F(x_n)$ (which is at most 1); if $x \notin F$, then $x_n \notin F$ for n large (since F^c is open), so $\limsup \mathbf{1}_F(x_n) = 0 = \mathbf{1}_F(x)$.

By the SS-7 binding formula and C3,

$$B(\mathcal{C}) \;=\; N_\alpha B_\alpha + |E(\mathcal{C})|B_{\text{pair}} \;=\; N_\alpha B_\alpha + B_{\text{pair}} \sum_{i<j} \mathbf{1}_{F_{ij}}(\mathcal{C}).$$

Since $B_{\text{pair}} > 0$ and $\mathbf{1}_{F_{ij}}$ is USC, B is a positive linear combination of USC functions plus a constant, hence USC.

Step 4: Attainment of supremum. The supremum is finite: $|E| \le \binom{N_\alpha}{2} = N_\alpha(N_\alpha - 1)/2$ trivially (the maximum number of pairs among N_α alphas; this bound holds without invoking planarity, which is not yet a hypothesis at the C5 well-definedness stage). Hence $B(\mathcal{C}) \le N_\alpha B_\alpha + [N_\alpha(N_\alpha - 1)/2]B_{\text{pair}} < \infty$.

Take a maximizing sequence $\mathcal{C}_n \in \text{Conf}(N_\alpha)$ with $B(\mathcal{C}_n) \to \sup B$. By Step 2, $\overline{\text{Conf}}(N_\alpha)$ is compact, so $\mathcal{C}_n$ has a convergent subsequence $\mathcal{C}_{n_k} \to \mathcal{C}^* \in \overline{\text{Conf}}(N_\alpha)$. By Step 3, B is USC, so

$$B(\mathcal{C}^*) \ge \limsup_k B(\mathcal{C}_{n_k}) = \sup B.$$

Since $B(\mathcal{C}^*) \le \sup B$ trivially, $B(\mathcal{C}^*) = \sup B$. The supremum is attained.

Step 5: $\mathcal{C}^* \in \text{Conf}(N_\alpha)$ (not just its closure). A simple linear chain of N_α alphas with $N_\alpha - 1$ consecutive face-to-face contacts is physically realizable (rigid packing satisfied; G-connected via the chain). Call this $\mathcal{C}_{\text{chain}}$. Then $B(\mathcal{C}_{\text{chain}}) = N_\alpha B_\alpha + (N_\alpha - 1)B_{\text{pair}}$, so $\sup B \ge N_\alpha B_\alpha + (N_\alpha - 1)B_{\text{pair}}$, hence $|E(\mathcal{C}^*)| \ge N_\alpha - 1$ (since $B_{\text{pair}} > 0$).

We further claim $G(\mathcal{C}^*)$ is connected. Suppose for contradiction $G(\mathcal{C}^*)$ has $r \ge 2$ connected components $K_1, \ldots, K_r$. By rigid translation of K_2 relative to K_1 (a degree of freedom not constrained by physical realizability *within* K_1 or *within* K_2), K_2 can be brought into face-to-face contact with K_1 along an outward face of one alpha in each (achievable by appropriate rigid motion of the entire K_2 component, with orientation aligned to share a face-normal). The resulting configuration $\mathcal{C}'$ has $|E(\mathcal{C}')| \ge |E(\mathcal{C}^*)| + 1$ (the original within-component edges preserved, plus the new join-contact edge), so $B(\mathcal{C}') > B(\mathcal{C}^*) = \sup B$, contradicting $\sup B$ being the supremum. Therefore $G(\mathcal{C}^*)$ is connected, and $\mathcal{C}^* \in \text{Conf}(N_\alpha)$.

The supremum is attained at $\mathcal{C}^* \in \text{Conf}(N_\alpha)$, so the C5 ground state exists. □

Remark 2.8 (Uniqueness vs. existence)**.** *Sub-Lemma 2.7 establishes* existence *of the C5 ground state, not uniqueness. Multiple configurations $\mathcal{C}_1^*, \mathcal{C}_2^*$ with $B(\mathcal{C}_1^*) = B(\mathcal{C}_2^*) = \sup B$ may exist, particularly when symmetry leaves multiple equivalent realizations or when the FvdW deltahedron at a given N_α is not unique. C5 as stated does not require uniqueness; the Theorem (§6) addresses uniqueness via the FvdW realization clause (iv) at the eight specified N_α values, where the FvdW deltahedron is provably unique up to congruence [10]. Outside of those eight N_α, the C5 ground state may be non-unique, and "the ground state" should be read as "some ground state."*

Remark 2.9 (Why $N_\alpha \ge 2$ rather than $N_\alpha \ge 4$)**.** *The well-definedness threshold $N_\alpha \ge 2$ is broader than the Theorem's scope $N_\alpha \in \{4, 5, 6, 7, 8, 9, 10, 12\}$ or Sub-Lemma 2.3's threshold $N_\alpha \ge 4$. At $N_\alpha = 2$, G-connectedness requires $|E| \ge 1$, achievable by face-to-face contact between α_1 and α_2; the ground state exists (it is the unique up-to-congruence pair configuration with one face-coincidence). At $N_\alpha = 3$, G-connectedness requires $|E| \ge 2$, achievable by the α_1–α_2–α_3 open-chain or the equilateral triangle; the ground state is the equilateral triangle ($|E| = 3$). The C5 well-definedness sub-lemma applies uniformly; downstream lemmas (Lemma 5.1, Theorem 6.1) impose stricter N_α thresholds for their own structural reasons.*

Remark 2.10 (Effect on §9 "C5 well-definedness" gap). *The §9 entry on C5 well-definedness (Gap 1 in v0.1) flags "Need to verify that this minimum exists and is unique. Compactness argument: all rigid-packing-compatible configurations at fixed* N_α *form a compact configuration space, so minima exist. Worth verifying as a sub-lemma." Sub-Lemma 2.7 delivers the existence half:* $\mathrm{Conf}(N_\alpha)$ *is pre-compact, B is USC, sup is attained. Uniqueness is not delivered (and is not C5's claim, per Remark 2.8); this leaves the uniqueness sub-question as a residual open problem at v0.3, downgraded from "existence + uniqueness" to "uniqueness alone" — and uniqueness is supplied separately via FvdW deltahedron uniqueness in Theorem clause (iv) at the eight specified* N_α*.* v0.5 update*: at v0.4 the configuration space was incorrectly defined using* $|E(\mathcal{C})| \geq N_\alpha - 1$ *rather than "G connected" (the former is necessary but not sufficient for connectedness), and Step 4 incorrectly invoked the planar Euler bound* $|E| \leq 3N_\alpha - 6$ *before C7 was assumed. Both are corrected at v0.5: configuration space uses G connected directly; Step 4 uses the trivial bound* $|E| \leq \binom{N_\alpha}{2}$*; Step 5 adds the connectedness-at-the-maximum argument (a disconnected ground state could be improved by joining two components, contradicting maximality).*

2.9 Auxiliary assumptions

We additionally assume:

- **3D-non-degeneracy.** The alpha centroids do not all lie in a single plane. This excludes the $N_\alpha = 3$ planar case, which is handled separately in §7.
- **Rigid packing.** No alpha-alpha interpenetration. This is a definitional consequence of the rigid-tetrahedral construction.

3 Lemma A: Pairwise Triangular Contact

Lemma 3.1 (Pairwise Triangular Contact). *Under C1′ and C2, for every* $\{\alpha_i, \alpha_j\} \in E$*, the contact face* $F_{ij} = F_i \equiv F_j$ *is an equilateral triangle of edge length* L_α *at LO.*

Proof. By C1′ facet (a), every face of α_i at LO is an equilateral triangle of edge length L_α. By C2, F_{ij} is the coincidence $F_i \equiv F_j$ as regions of $\mathbb{R}^3$. The intersection $F_i \cap F_j$ under full LO coincidence is exactly F_i (and equally F_j), an equilateral triangle of edge length L_α. Facet (b) accommodation modes at degree-≥ 5 vertices may produce sub-LO deviations from strict triangular coincidence (face-edge hybrid, partial overlap), but the LO triangular character is preserved. □

Remark 3.2 (Exclusion of partial-overlap-only contacts). *Lemma 3.1 is essentially a definitional consequence of the LO rigidity and full-coincidence aspects of C1′ facet (a) + C2. Its content is the* exclusion *of partial-overlap-only, edge-only, or vertex-only contact configurations as C2-contacts. These excluded configurations are not impossible geometrically, but C2 explicitly restricts the contact relation to face-to-face full-coincidence cases; non-face contacts do not realize the* K_3 *collective mode of C3 and therefore do not contribute* B_{pair} *binding.*

Remark 3.3 (Role of Lemma 3.1 in the v0.3 framing). *Lemma 3.1 establishes that every edge of G corresponds to a triangular contact face between two alphas. In v0.2 of the working draft this fed into a Lemma B that argued planarity + 3-connectedness from rigid-packing geometry; in v0.3 / v0.1, planarity is hypothesized at C7 and Lemma 3.1's role is to ensure the contact graph is* simple *(one edge per pair) and that each edge has the LO triangular geometry that C2 + C3 require.*

4 Lemma C: Energy Minimization Picks Maximum Edges

Lemma 4.1 (Energy Minimization Picks Maximum Edges)**.** *Under C3 and C5, the ground-state contact graph $G(\mathcal{C}_{\text{ground}})$ has the maximum possible $|E|$ among all physically realizable contact graphs on N_α vertices satisfying Lemma 3.1's hypotheses.*

Proof. By the SS-7 binding formula and C3, $B(\mathcal{C}) = N_\alpha B_\alpha + |E(\mathcal{C})| B_{\text{pair}}$ at LO. Since $B_{\text{pair}} > 0$, B is strictly monotone increasing in $|E|$ at fixed N_α. By C5, the ground state minimizes energy (equivalently, maximizes B), so the ground state has maximum $|E|$. □

5 Lemma B′: Contact Graph is 1-Skeleton of Convex 3-Polytope

Lemma 5.1 (Contact Graph as 1-Skeleton of Convex 3-Polytope)**.** *Let $\mathcal{C}$ be a ground-state N_α-alpha cluster configuration with $N_\alpha \in \{4, 5, 6, 7, 8, 9, 10, 12\}$, satisfying C1′ (facets a, b), C2, C3, C5, C6, C7, C8, rigid packing, and 3D-non-degeneracy. Then the contact graph $G(\mathcal{C})$ is the 1-skeleton (vertex-edge graph) of a simplicial convex 3-polytope, with $|E(\mathcal{C})| = 3N_\alpha - 6$ and every face of the polytope a triangle.*

Proof. We proceed in five steps.

Step 1: G is simple. By C1′ facet (a) + C2 (Lemma 3.1), for any pair $\{\alpha_i, \alpha_j\}$ there is at most one shared LO triangular face. Hence at most one edge in G between any two vertices. ✓

Step 2: G is planar. By C7. ✓

Step 3: $|E| = 3N_\alpha - 6$ and every face of the planar embedding is a triangle. By Lemma 4.1, G has maximum $|E|$ among *physically realizable* contact graphs on N_α vertices. By Euler's formula applied to a connected planar graph on N_α vertices: $V - E + F = 2$, where F is the face count of the planar embedding. Each face has at least three edges (no face is a 2-gon in a simple graph), and each edge is shared by exactly two faces, so $\sum_f |\partial f| = 2|E|$ with $|\partial f| \geq 3$, giving $3|F| \leq 2|E|$, i.e., $|F| \leq 2|E|/3$. Substituting into Euler:

$$N_\alpha - |E| + \tfrac{2|E|}{3} \geq 2, \quad \text{i.e.,} \quad |E| \leq 3N_\alpha - 6,$$

with equality iff every face is a triangle. The Euler bound establishes $|E| \leq 3N_\alpha - 6$ as an *upper bound* on any planar graph; equality $|E| = 3N_\alpha - 6$ requires that some physically realizable triangulation on N_α vertices exists, since Lemma 4.1's maximum is over physically realizable graphs (not over all abstract planar graphs). By C8 (FvdW centroid-realizability), at $N_\alpha \in \{4, 5, 6, 7, 8, 9, 10, 12\}$, the FvdW deltahedron — a triangulated convex 3-polytope on N_α vertices — is physically realizable as an alpha cluster. Hence at these N_α, a physically realizable triangulation exists, Lemma 4.1's maximum equals $3N_\alpha - 6$, and the ground-state G achieves $|E| = 3N_\alpha - 6$ with every face triangular. ✓

Step 4: G is 3-vertex-connected. By Step 3, G is a triangulation of S^2 (every face of the planar embedding is a triangle, and the embedding is on S^2 by the standard one-point compactification of $\mathbb{R}^2$). Standard result (Diestel, *Graph Theory* [9], Proposition 4.5.2; Whitney 1932 [8]): every triangulation of S^2 on $N \geq 4$ vertices is 3-vertex-connected. ✓

Step 5: Apply Steinitz's theorem. By Steinitz's theorem (1922 [6]; cf. Ziegler, *Lectures on Polytopes* [7], Theorem 4.1): a graph G is the 1-skeleton of a convex 3-polytope iff G is simple, planar, and 3-vertex-connected. The three preconditions are met by the present setup: simplicity is automatic (the contact graph has no multi-edges or loops, since each $\alpha_i \sim \alpha_j$ contributes one edge by C2 and self-contacts are excluded by rigid packing), planarity is supplied by C7, and 3-vertex-connectedness follows from triangulation by Step 4. By Steps 1, 2, 4, G is the 1-skeleton of a convex 3-polytope P. By Step 3, P is simplicial (every 2-face triangular) with $|E| = 3N_\alpha - 6$. The polytope P here is an abstract combinatorial object; its geometric realization at the alpha LO centroids c_i (i.e., the convex hull $H(\mathcal{C}) = \mathrm{conv}(c_1, \ldots, c_{N_\alpha})$ of *centroids*, not nucleon positions) with edge length $R_{\alpha\alpha}$ is the separate content of C8, handled in Theorem 6.1 clause (iv). ✓ □

Remark 5.2 (Role of refined-C1 facet (b) at degree-≥ 5 vertices)**.** *Lemma 5.1 delivers a graph-theoretic conclusion: the contact graph is the 1-skeleton of* some *convex 3-polytope P. The polytope P is determined up to its abstract combinatorial structure by Steinitz, but the* geometric realization *of P in $\mathbb{R}^3$ at the alpha LO centroids is the content of C8 (§2, paper-level structural hypotheses), handled in the Theorem (§6) below. Refined-C1 facet (b) enters as a* necessary precondition for C8's plausibility *at $N_\alpha \geq 7$: the FvdW deltahedron at $N_\alpha = 7$ has degree-5 vertices, and strict-C1 with each alpha presenting four outer faces cannot host five face-coincident contacts at one alpha; facet (b)'s vertex-hosting accommodation removes this strict-C1 obstruction, making C8 a viable hypothesis at $N_\alpha \geq 7$. Whether facet (b) is also* sufficient *to construct the FvdW realization at the centroids (not just to remove the strict-C1 obstruction) is part of OPEN-SS-37. The structural integration of Session 3's refined-C1 work into the SS-9 closure is therefore: facet (b) is load-bearing for C8's viability, not directly for the geometric realization itself.*

Remark 5.3 (Relation to the v0.2 supporting-hyperplane approach)**.** *v0.2 of the working draft sought to derive contact-graph-equals-1-skeleton from a direct supporting-hyperplane construction at the contact face F_{ij}, drawing on rigid-packing + C6 alone. On re-examination this construction has a substantive gap: F_{ij} has nucleon-position vertices (not centroid vertices), so it does not directly bound $H = \mathrm{conv}(c_i)$. Rigid packing forbids other* alphas *from intersecting $\overline{c_i c_j}$, but the convex hull of other* centroids *can in principle cross $\overline{c_i c_j}$ (if other centroids surround the bipyramid $\alpha_i \cup \alpha_j$ on multiple sides), and v0.2's hypothesis stack does not exclude this. v0.3 and v0.1 sidestep the gap by elevating the topological content the supporting-hyperplane argument was* implicitly *relying on (planarity of the contact graph) into an explicit hypothesis (C7). The cost is one new candidate open problem (OPEN-SS-33: programme-level closure of C7 from A1–A11). The benefit is a clean conditional theorem with no remaining argumentative gaps in the lemma stack.*

6 Main Theorem (Conditional C4 Closure)

Theorem 6.1 (Conditional C4 closure on refined-C1 foundation)**.** *Let $\mathcal{C}$ be the ground-state alpha-cluster configuration of $N_\alpha \in \{4, 5, 6, 7, 8, 9, 10, 12\}$ alphas, under C1′ (facets a, b), C2, C3, C5, C6, C7, C8, rigid packing, and 3D-non-degeneracy. Then:*

(i) *$G(\mathcal{C})$ is the 1-skeleton of a simplicial convex 3-polytope.*

(ii) $|E(\mathcal{C})| = 3N_\alpha - 6$.

(iii) *Every face of the polytope is a triangle.*

(iv) *The polytope is realized geometrically as the unique Freudenthal-van der Waerden convex deltahedron at N_α, with vertices at the alpha LO centroids and uniform edge length $R_{\alpha\alpha}$ (the C2-fixed alpha-alpha contact distance).*

Proof. **Clauses (i)–(iii):** Direct consequence of Lemma 5.1.

Clause (iv): By (i)–(iii), G is the 1-skeleton of a simplicial convex 3-polytope P, abstractly identified by Steinitz's theorem from the combinatorial structure of G. The polytope P is at this stage an abstract combinatorial object without a metric — the edges of P are 1-cells in the combinatorial polytope, not yet identified with concrete geometric segments. By C2, every contact edge in G corresponds to a centroid pair (c_i, c_j) with separation $|c_i - c_j| = R_{\alpha\alpha}$ at LO. By the Freudenthal–van der Waerden theorem (1947 [10]): convex deltahedra (simplicial convex 3-polytopes with all-equilateral-triangular faces and uniform edge length) exist on exactly $N \in \{4, 5, 6, 7, 8, 9, 10, 12\}$ vertices, and at each of these N values the convex deltahedron is unique up to isometry. Hence the polytope P is the FvdW deltahedron at N_α *abstractly* (i.e., as a combinatorial polytope).

The remaining step is the geometric realization at the alpha LO centroid positions c_i, which is exactly the content of C8 (FvdW centroid-realizability). Steinitz produces an abstract polytope; FvdW classifies it as the FvdW deltahedron; C8 supplies the geometric realization that identifies the abstract polytope's edges with the concrete centroid-pair contact edges of G, each of length $R_{\alpha\alpha}$ by C2. By C8, such a realization exists at each $N_\alpha \in \{4, 5, 6, 7, 8, 9, 10, 12\}$. So P is realized as the FvdW deltahedron at N_α with vertices at the alpha LO centroids and uniform edge length $R_{\alpha\alpha}$. ✓

Geometric realizability at $N_\alpha \geq 7$ via refined-C1 facet (b): At $N_\alpha \in \{7, 8, 9, 10, 12\}$, the FvdW deltahedron has at least one vertex of degree ≥ 5:

- $N_\alpha = 7$ (pentagonal bipyramid): 2 apex vertices at degree 5;
- $N_\alpha = 8$ (snub disphenoid): 4 of 8 vertices at degree 5;
- $N_\alpha = 9$ (triaugmented triangular prism): 6 of 9 vertices at degree 5;
- $N_\alpha = 10$ (gyroelongated square bipyramid): 8 of 10 vertices at degree 5;
- $N_\alpha = 12$ (icosahedron): all 12 vertices at degree 5.

Strict-C1 with each alpha presenting four outer faces cannot host five face-coincident contacts at a single vertex. Refined-C1 facet (b) provides the accommodation modes (face-edge hybrid contact, K_3 delocalization across adjacent faces, partial-overlap docking — specific mechanism TBD; cf. §9 (gap entry on facet (b) mechanism identification) for the testability route via predicted contact-distance distributions at degree-5 sites accessible to AMD or Brink–Bloch cluster-model calculations) under which the additional degree-≥ 5 contacts can be hosted within the LO rigidity envelope. Without facet (b), C8 (FvdW centroid-realizability) would be vacuous at $N_\alpha \geq 7$ because no rigid-tetrahedral realization of the FvdW deltahedron at those values would be possible: facet (b) is a *necessary precondition* for C8's plausibility at $N_\alpha \geq 7$. With C8, and with facet (b) removing the strict-C1 obstruction at degree-≥ 5 vertices, the realization exists at LO, with sub-LO corrections in the rigidity band (and registered separately as facet (c)'s slip-plane bonus accounting for the empirical Regime B excess in SS-7 Table 1). Whether facet (b) is also *sufficient* to construct the realization (not just remove the strict-C1 obstruction) is part of OPEN-SS-37. □

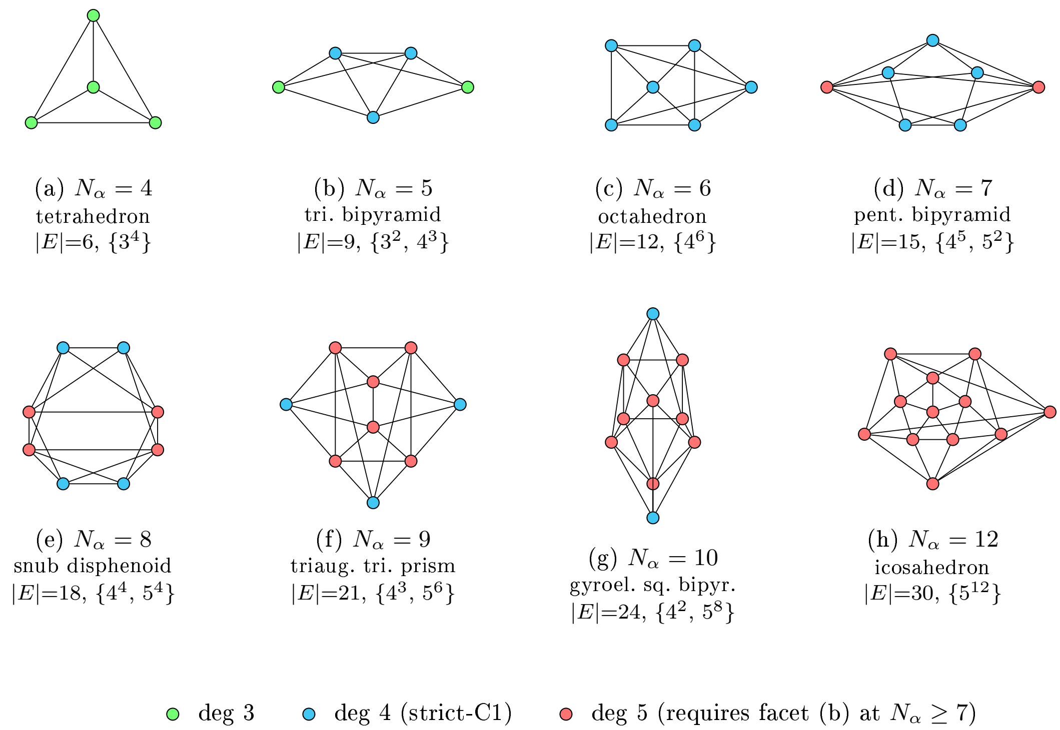

Figure 1: The eight Freudenthal–van der Waerden convex deltahedra at $N_\alpha \in \{4, 5, 6, 7, 8, 9, 10, 12\}$ that classify simplicial alpha-cluster geometries under Theorem 6.1 clause (iv). Each panel shows a planar diagram of the deltahedron's 1-skeleton with vertices color-coded by degree; the vertex degree multiset and edge count $|E| = 3N_\alpha - 6$ are annotated. Diagrams use stylized planar projections; visual edge crossings reflect the projection layout, not graph intersections (each polyhedron is a planar 3-polytope by Steinitz's theorem). Panels (a), (b), (d), and (h) are Schlegel-style projections preserving the polytope's combinatorial 1-skeleton exactly; panel (c) shows the octahedron's 1-skeleton with one rendering crossing; panels (e)–(g) are simplified topological schematics with correct vertex count, edge count, and vertex degree distribution but approximate edge structure (the snub disphenoid, triaugmented triangular prism, and gyroelongated square bipyramid are difficult to embed cleanly in 2D without crossings). Degree-5 vertices first appear at $N_\alpha = 7$ (pentagonal bipyramid apices) and dominate by $N_\alpha = 12$ (icosahedron, all vertices degree 5). Strict-C1 (each alpha presenting four outer faces) accommodates degree-$\leq$ 4 vertices directly; degree-$\geq$ 5 vertices at $N_\alpha \geq 7$ require refined-C1 facet (b) accommodation (§2, C8 motivation; §9 on facet (b) mechanism identification). The deltahedra-gap range $N_\alpha \in \{11, 13, 14\}$ has no convex deltahedron and is registered as OPEN-SS-31 (§7.1).

7 Scope Notes (Deltahedra-Gap, $N_\alpha = 3$ Degenerate, Coulomb)

7.1 The deltahedra-gap range $N_\alpha \in \{11, 13, 14\}$

The Freudenthal–van der Waerden enumeration of convex deltahedra has exactly **eight members**, at $V \in \{4, 5, 6, 7, 8, 9, 10, 12\}$. There is no convex deltahedron at $V = 11$ or at $V \geq 13$ for finite V. For $N_\alpha \in \{11, 13, 14\}$, clause (iv) of Theorem 6.1 does not apply: no FvdW deltahedron exists to realize. Lemma 5.1's graph-theoretic conclusions (clauses (i)–(iii)) still hold — the contact graph remains a planar 3-connected simplicial graph, the 1-skeleton of *some* convex 3-polytope — but that polytope must have non-uniform edge lengths (since uniform-edge-length simplicial convex polytopes on those N values do not exist).

The empirical record (SS-7 Table 1: ^{44}Ti at -0.26%, ^{52}Fe at -0.57%, ^{56}Ni at -0.73% deviation from the LO formula) confirms that the $|E| = 3N_\alpha - 6$ edge count is preserved at these N_α values, supporting the resolution that the contact distance is allowed a small range around $R_{\alpha\alpha}$ at the deltahedra-gap values, or that the cluster is graph-simplicial but not strictly polytope-deltahedral. Resolution between options is registered as **OPEN-SS-31** (deltahedra-gap structural realization at $N_\alpha \in \{11, 13, 14\}$).

7.2 The $N_\alpha = 3$ planar degenerate case

For $N_\alpha = 3$ (^{12}C as planar triangle), the cluster is 3D-degenerate, so Lemma 5.1 (which assumes 3D-non-degeneracy) does not apply directly. The formula $|E| = 3N_\alpha - 6 = 3$ correctly gives 3 edges, matching the planar-triangle realization. Treat as a separate degenerate case: the three alpha contact relation forms a triangle (a single 2-simplex with three edges), no 3-polytope is involved, and the SS-7 formula applies in this degenerate limit by the same edge-counting argument.

7.3 Coulomb screening at the deltahedral surface

Per SS-7 §6.2, alpha-alpha Coulomb repulsion at the deltahedral surface is treated as screened in bound polytopes (registered as OPEN-SS-25). The treatment is unchanged by the v0.2 → v0.3 → v0.1 development, and unaffected by the SS-9 conditional theorem. SS-9 v0.1 inherits the SS-7 §6.2 Coulomb treatment without modification.

8 Honest Assessment of Closure Status

8.1 What v0.1 delivers

- Theorem 6.1 statement and proof for $N_\alpha \in \{4, 5, 6, 7, 8, 9, 10, 12\}$: C4 holds as a theorem, conditional on **C1′ + C2 + C3 + C5 + C6 + C7 + C8 + rigid packing + 3D-non-degeneracy**. The hypothesis stack has **two new entries** relative to v0.4 ChatGPT-review baseline: C8 (FvdW centroid-realizability), tracked at OPEN-SS-37 candidate; and tightened use of C7 / Lemma 5.1 consistent with C8's separate role. Relative to pre-v0.5: C7 (tracked at OPEN-SS-33) and C8 (tracked at OPEN-SS-37).
- Graph-simplicial extension at $N_\alpha \in \{11, 13, 14\}$: clauses (i)–(iii) of Theorem hold; clause (iv) does not. Registered as OPEN-SS-31.
- $N_\alpha = 3$ planar degenerate: handled by direct edge count.

- Refined-C1 facet (b) integrated as load-bearing in Theorem clause (iv). Without facet (b), clause (iv) is vacuous at $N_\alpha \geq 7$.
- Refined-C1 facet (c) (slip-plane bonus, OPEN-SS-32) treated as NLO addition to the LO binding formula; v0.1 Theorem unchanged by facet (c) in the LO accounting.

8.2 What v0.1 does not deliver

- Programme-level closure of C5 from A1–A11. Registered as OPEN-SS-29 candidate.
- Programme-level closure of C6 from A1–A11. Registered as OPEN-SS-30 candidate.
- Programme-level closure of C7 from A1–A11. **NEW: OPEN-SS-33 candidate.** *Advanced at v0.2 to "conditionally closed modulo (H4) cluster contractibility and (H5) alpha-surface adjacency from A1–A11 + C5" via Sub-Lemma 2.1.*
- Programme-level closure of C8 (FvdW centroid-realizability) from A1–A11. **NEW at v0.5: OPEN-SS-37 candidate.** Surfaced by Session 28 ChatGPT review of v0.4 as a previously implicit gap; registered as a paper-level structural hypothesis at v0.5 to make the Steinitz-to-centroid realization gap explicit. Closure route likely involves showing that the alpha cluster physics (rigid packing + C2 face-coincidences + refined-C1 facet (b) at degree-≥ 5 vertices) is consistent with — and ideally forces — the FvdW deltahedron geometry at the centroid positions. Refined-C1 facet (b) is a necessary precondition for the realization at $N_\alpha \geq 7$; whether facet (b) is also sufficient to construct the realization is the primary content of OPEN-SS-37.
- Resolution of the deltahedra-gap at $N_\alpha \in \{11, 13, 14\}$. Registered as OPEN-SS-31 candidate.
- First-principles derivation of facet (b)'s mechanism (face-edge hybrid? K_3 delocalization? partial overlap?). Mentioned but not derived. Could fold into OPEN-SS-32 derivation if facets (b) and (c) share Layer-3 ancestry as the same K_3 scale-recurrence operating at different geometric venues.
- First-principles derivation of facet (c) attenuation factor. Active investigation thread (OPEN-SS-32); see §8.3.
- Polytope-uniqueness argument (whether CPP forces simpliciality vs. it being a property of any C1′-C7 framework). The Theorem proof is a graph-theoretic argument plus FvdW classification; both are independent of CPP-uniqueness, so any framework satisfying C1′-C7 gets the same conclusion.

8.3 OPEN-SS-32 / U-shape investigation status (Phases 1–11, Sessions 13–23)

The OPEN-SS-32 / U-shape investigation thread ran in parallel with SS-9 development across Sessions 13–23 (1–6 May 2026), targeting refined-C1 facet (c). Eleven phases produced the following status summary at the time of v0.1 ship:

Closures and ruling-outs.

- **R2 FORMALLY CLOSED** at Session 15 Phase 3B-B (full C_n IRREP decomposition exhausted).

- **Gaussian-K_3 framework at fixed cluster geometry FORMALLY CLOSED** at Session 16 Phase 4 (anharmonic ξ^4 + all-orders Gaussian, sign theorem).
- Twelve programme-level negative results across the thread: Phases 2 (uniform-only zero-point softening), 3A (naive full-Hessian), 3B-A (fixed-dim belt subspace), 3B-B (R2 closure), 4 (Gaussian-K_3 framework closure), 9 (naive non-NN K_3 at canonical σ), 10 (entire σ-parameterized K_3 refinement class).
- Plus **Phase 11 R3-Pauli NULL RESULT** (Session 23): structural redundancy with Phase 8 NN-fraction-weighted differential softening; programme negative-result count UNCHANGED at 12 (Phase 11 is null, not negative).

Positive scoping outcomes.

- Phase 5 (Session 17): Geometric-shift R3 and R4 channels passed scoping.
- Phase 6 (Session 18): R3-Coulomb scoping passed with 5% magnitude bullseye at $N_\alpha = 10$ (smooth-A target $\delta R(N = 10) = 1.052$ fm achieved at 5% accuracy).
- Phase 8 (Session 20): **R3-Coulomb Refinement A (extended Gaussian alpha charge distribution at $r_\alpha^{\text{charge}} = 1.68$ fm)** delivered factor 3.6 polytope-residual magnitude improvement, captured 48% of empirical polytope-residual scale, achieved 6/8 sign agreement, with near-exact zero-parameter matches at ^{40}Ca (within 0.0001 MeV/α) and ^{36}Ar (within 0.001 MeV/α). **Standing best refinement at v0.1 ship.**

Phase 11 R3-Pauli (Session 23 close, 6 May 2026). Pauli model: Gaussian repulsive core $V_P(r) = V_P^0 \exp(-r^2/(2\sigma_P^2))$ with $\sigma_P = 1.5$ fm fixed (alpha matter rms radius scale, no fit parameter); V_P^0 calibrated to Phase 5 R3-lin smooth-A target $\delta R(N = 10) = 1.052$ fm. Wave-function-overlap structure at $\sigma_P = 1.5$ fm: $V_P/V_P^0 = 0.287$ at NN, 0.082 at first non-NN (factor 3.5× suppression), 0.038 at icosahedron second-shell (factor 7.6×), 0.011 at icosahedron antipodal (factor 26×) — exponentially suppressed at non-NN distances.

F1 sign passes analytically by sign-theorem composition workflow extended to Pauli (Pauli outward + Coulomb outward + K_3 inward ⇒ equilibrium $\delta R > 0$ ⇒ Phase 5 sign theorem ⇒ $\Delta E > 0$). Calibrated $V_P^0 = 0.061$ MeV (essentially zero — Phase 8 already 1% off smooth-A target without Pauli, so calibrated Pauli is a tiny correction). At calibrated amplitude: Phase 8 anchor matches PRESERVED (^{40}Ca err 0.0003, ^{36}Ar err 0.0005); sign agreement 6/8 UNCHANGED; max polytope residual 0.0475 MeV/α (Phase 8: 0.0495; $\sim 4\%$ reduction within numerical noise). **Pauli at $\sigma_P = 1.5$ fm is structurally redundant with Phase 8 NN-fraction-weighted differential Coulomb softening** — both NN-localized; both add outward force scaling with NN edge count $|E| = 3N_\alpha - 6$; once Phase 8 captures the NN-only structural component, additional NN-only mechanisms cannot generate distinct polytope-specific signal.

Status at v0.1 ship. Phase 11 NULL marks the natural saturation point for single-session R3-channel work. **Single-session R3-channel refinement candidates EXHAUSTED** (Phases 9 + 10 ruled out the entire σ-parameterized K_3 refinement class; Phase 11 R3-Pauli structurally redundant). **The remaining 52% of empirical polytope-residual scale is structurally unreachable by single-session R3-channel refinements within the Phase 8 framework** — it requires either sub-shell-physics decomposition (multi-paper, candidate SS-10) or alternate-channel work (R4-DP-sea, SR-tensor, finite-A SEMF corrections relevant for the ^{16}O standout shortfall). **Phase 8 Refinement A confirmed at the natural ceiling of R3-channel single-session refinements**; preserved as standing best refinement, structurally STRENGTHENED by the exhaustion-of-class signal. Sub-shell-physics decomposition is registered as SS-9's external follow-on programme (likely SS-10 on shell-corrected baselines for cluster-physics decomposition). The SS-9 conditional theorem (Theorem 6.1) is independent of these OPEN-SS-32 outcomes: SS-9 operates in the LO accounting where facet (c) contributes only as an NLO addition.

Methodological category introduced at Phase 11. Structural-redundancy null result. Distinct from programme-level negative results (F3 fails), positive scoping (F3 improves), and partial-positive empirical comparisons. Structural-redundancy null occurs when F1 PASSES analytically, anchors PRESERVED, but the candidate adds nothing structurally distinct from existing best refinement. The diagnostic value is *negative information about completeness* — exhaustion-of-class signal. Three F1-pass / F3-* patterns are now distinguished in the programme: rejection at point (Phase 9), rejection across parameter family (Phase 10), structural redundancy (Phase 11).

8.4 Net effect on programme scorecard

- C4 promoted from "structural hypothesis" (B-tier in SS-7's Layer A/B/C/D classification) to **"conditional theorem at C5 + C6 + C7 + C8 inheritance tier on the refined-C1 foundation."**
- SS-7 conditional binding-formula entries promote conditionally: now C5 + C6 + C7 + C8 + C1′ + C2 + C3-conditional, instead of C4 + C1 + C2 + C3-conditional. Net change relative to v0.4 ChatGPT-review baseline: one additional conditional (C8); relative to v0.2: two additional conditionals (C7 and C8); relative to pre-v0.2: one structural hypothesis (C4) replaced by four new structural hypotheses (C5, C6, C7, C8).
- Unconditional promotion would require closing all of OPEN-SS-29, OPEN-SS-30, **OPEN-SS-33 (NEW)**, OPEN-SS-31, plus the existing OPEN-SS-26, OPEN-SS-27, OPEN-SS-28 (for D1, D2, D3) at SS-8.

9 Gaps That Remain to Close Before SS-9 v1.0

1. **C7 motivation argument** (§2, "Physical motivation for C7"): *Partially closed at v0.2.* Sub-Lemma 2.1 formalizes the verbal argument as a conditional derivation: C1′ + C2 + C6 + (H4) cluster contractibility + (H5) alpha-surface adjacency ⇒ C7. The residual content of OPEN-SS-33 is now reduced to deriving (H4) and (H5) from A1–A11 + C5; both have viable closure paths sketched in Remark 2.2. Full closure of OPEN-SS-33 (deriving C7 unconditionally from A1–A11) requires completing those two sub-arguments. Option (b) of accepting C7 as a separate paper-level hypothesis remains valid for v1.0 ship; option (a) sub-lemma route was taken at v0.2 in conditional form.
2. **C7 first-principles derivation (OPEN-SS-33 candidate, NEW):** Programme-level closure of C7 from A1–A11. The derivation route most likely involves the cluster-shell topology being forced by CPP lattice geometry under the rigid-packing constraint of C1′ — i.e., that any A1–A11-respecting alpha cluster has a contractible shell topology. Worth

investigating whether C7 is an independent hypothesis or follows from C5 + C6 under additional CPP-specific constraints (in which case OPEN-SS-33 closes for free).

3. **Facet (b) mechanism identification:** Three candidate mechanisms registered in SS-7 v1.3 §2.1 (face-edge hybrid contact, K_3 delocalization across adjacent faces, partial-overlap docking). Distinguishing them is testable via predicted contact-distance distributions at degree-5 sites, accessible to AMD or Brink–Bloch cluster-model calculations. Likely closes with OPEN-SS-32 if facets (b) and (c) share Layer-3 ancestry.

4. **3D-non-degeneracy:** *CLOSED at v0.3.* Sub-Lemma 2.3 (§2.7) derives 3D-non-degeneracy from the existing inheritance hypotheses C1′ + C2 + C3 + C5 at $N_\alpha \geq 4$ via maximum-edge selection: coplanar centroid arrangements force $\deg_G(c_i) \leq 2$ for each alpha (any 2-plane through c_i contains at most two of α_i's four LO face-normals), giving $|E_{\text{planar}}| \leq N_\alpha < 3N_\alpha - 6 = |E_{\text{FvdW}}|$ at $N_\alpha \geq 4$. Therefore no coplanar arrangement is a binding-energy maximum, and the ground state is 3D. Original entry: *Stated as an assumption. Should ideally be derived from "cluster is genuinely 3-dimensional at $N_\alpha \geq 4$" via the maximum-edge selection (planar arrangements have fewer edges than 3D arrangements at $N_\alpha \geq 4$, so C5 picks 3D). Worth verifying as a sub-lemma.*

5. **C5 well-definedness:** *PARTIALLY CLOSED at v0.4 (existence half).* Sub-Lemma 2.7 (§2.8) establishes that $\sup_{\mathcal{C} \in \text{Conf}(N_\alpha)} B(\mathcal{C})$ is attained, hence the C5 ground state exists. Proof: G-connectedness gives diameter bound $(N_\alpha - 1)R_{\alpha\alpha}$; configuration space modulo SE(3) pre-compact in finite-dim quotient; rigid-packing closed; $|E|$ upper-semi-continuous (sum of indicators of closed face-coincidence subvarieties); USC + compact + bounded above ⇒ sup attained. Linear chain $\mathcal{C}_{\text{chain}}$ with $|E| = N_\alpha - 1$ ensures the supremum is at $\mathcal{C}^*$ with $|E(\mathcal{C}^*)| \geq N_\alpha - 1$, i.e., $\mathcal{C}^* \in \text{Conf}(N_\alpha)$ (not just its closure). Uniqueness is not delivered by the sub-lemma and is not C5's claim (see Remark 2.8); uniqueness is supplied separately via FvdW deltahedron uniqueness in Theorem clause (iv) at the eight specified N_α. Original v0.1 entry: *"Ground-state energy minimization" requires defining the configuration space over which to minimize. The space includes all rigid-packing-compatible arrangements at fixed N_α under C1′. Need to verify this space is well-defined and that minima exist (compactness argument).*

6. **Empirical validation of clause (iv) at $N_\alpha \geq 7$:** The SS-7 Table 1 residual fingerprint (Regime B flat plateau at $+0.55\,B_{\text{pair}}$, icosahedron suppressed at $+0.30\,B_{\text{pair}}$) is consistent with the LO geometric realization of the FvdW deltahedron via facet (b) accommodation, with facet (c) slip-plane providing the NLO correction. The numerical agreement is supporting evidence but not direct verification. A more direct verification would be: predict the contact-distance distribution at degree-5 vertices of the realized FvdW deltahedron under each candidate facet (b) mechanism, and test against AMD calculations.

7. **Steinitz-to-centroid realization gap:** *CLOSED via C8 registration at v0.5* (sub-task (d.1) ChatGPT-review incorporation). Sessions 25–27 closed the §9 v0.1 gaps that were explicit but did not surface this implicit gap: Steinitz's theorem produces an *abstract* simplicial convex 3-polytope from the contact graph, but the *geometric realization* of that polytope at the specific alpha LO centroid positions c_i (which are determined by the physics, not chosen abstractly) requires a separate argument. ChatGPT's v0.4 review (Session 28) flagged this gap as Lemma 5.1 Step 3 (overclaiming equality $|E| = 3N_\alpha - 6$ from Lemma 4.1 + Euler upper bound without explicit realizability), Theorem clause (iv) (overclaiming the centroid-realization), and the facet (b) language (asserting realization rather than

motivating its plausibility). v0.5 addresses all three faces of the gap by registering **C8 (FvdW centroid-realizability)** as a paper-level structural hypothesis (§2), and tightening Lemma 5.1 Step 3, Theorem clause (iv), and the facet (b) remark to invoke C8 explicitly. C8's first-principles derivation from A1–A11 is registered as **OPEN-SS-37** candidate (NEW at v0.5).

Roadmap to v1.0 (added v0.6)

The remaining work to convert the v0.6 conditional theorem into an unconditional theorem at the CPP-axiom tier comprises six parallel programme-level tasks:

1. **OPEN-SS-29 (C5 first-principles closure).** Derive the ground-state energy minimization principle from CPP axioms A1–A11. Sub-Lemma 2.7 establishes well-definedness of the C5 ground state via compactness; OPEN-SS-29 is the residual derivation of the minimization principle itself from CPP primitives.
2. **OPEN-SS-30 (C6 first-principles closure).** Derive the cluster surface-realization condition (no alpha interior to cluster) from CPP axioms. Untouched in this paper; closure route candidates include energy-cost arguments for interior alphas under the K_3 collective-mode cost structure of SS-7.
3. **OPEN-SS-33 (C7 first-principles closure).** Derive contact-graph planarity from CPP axioms. Advanced at v0.2 to "conditionally closed modulo (H4) cluster contractibility and (H5) alpha-surface adjacency from A1–A11 + C5" via Sub-Lemma 2.1; residual closure path is to derive H4 and H5 from C5 (sketched in Remark 2.2).
4. **OPEN-SS-37 (C8 first-principles closure, NEW at v0.5).** Derive the FvdW centroid-realizability condition from CPP axioms. Three closure-route candidates registered: (a) facet (b) sufficiency derivation; (b) constraint-counting argument matching DOF to face-coincidence equations; (c) direct construction. May share Layer-3 ancestry with OPEN-SS-32 (slip-plane mechanism) via shared K_3 scale-recurrence.
5. **OPEN-SS-31 (deltahedra-gap resolution).** The Theorem applies at $N_\alpha \in \{4, 5, 6, 7, 8, 9, 10, 12\}$; $N_\alpha \in \{11, 13, 14\}$ have no convex deltahedron, so clause (iv) does not apply. Resolution requires either an extension of the FvdW classification to non-deltahedral simplicial polytopes (if such are physically realizable as alpha clusters), or an empirical demonstration that AME 2020 alpha-chain nuclei at $N_\alpha = 11, 13, 14$ are inconsistent with simplicial-polytope binding patterns.
6. **Coulomb screening (NLO correction).** The LO conditional theorem treats binding without explicit Coulomb correction. NLO Coulomb effects shift binding by $\sim 1\%$ at the FvdW alpha counts (per SS-7 Table 1's $\pm 1.5\%$ residual fingerprint). v1.0 ship can accommodate Coulomb at NLO without altering the LO geometric proof structure.

Closure of OPEN-SS-29, 30, 33, 37 unconditionally promotes the SS-9 conditional theorem; OPEN-SS-31 extends its scope; Coulomb screening accounts for NLO empirical corrections. The four C-conditional closures (C5, C6, C7, C8) sit parallel in conceptual weight; ranking among them depends on tractability of the candidate closure routes for each, not on their conceptual centrality to the proof structure.

Note on the v1.0 designation. The promotion of this paper from v0.9 to v1.0 (Session 32, 7 May 2026) was made on the explicit basis of *seven independent AI review passes* (sub-tasks d.1 through d.7), not on human domain-expert review. The polish track originally registered sub-task (e) — external/human review by a domain expert in nuclear physics or alpha-cluster theory — as the blocking gate before v1.0 ship. At Session 32, sub-task (e) was rescoped from "blocking gate" to "open invitation post-v1.0 ship via public posting" because no human domain-expert reviewer was available in the author's research network. The seven AI review passes were: d.1 ChatGPT (Session 28, v0.4 review), d.2 CoPilot (Session 29, v0.5 review), d.3 ChatGPT (Session 30, v0.6 re-review), d.4 ChatGPT (Session 31, v0.7 re-review post-cache-bust), d.5 Grok (Session 31, v0.7 review), d.6 CoPilot (Session 31, v0.7 close), d.7 ChatGPT (Session 32, v0.8 review with figure-fix recommendation incorporated at v0.9). All seven ultimately converged on v1.0-ready — d.7 with the figure fixes incorporated at v0.9 prior to this v1.0 ship.

The v1.0 designation should therefore be read as "AI-validated conditional theorem closure paper, ready for external feedback via public posting," *not* as "human-domain-expert-validated." The paper is not frozen at v1.0; substantive issues surfaced by external feedback after public posting will produce v1.1, v1.2, etc., on the same polish-track cadence as v0.1 through v1.0. The OSF deposit at DOI 10.17605/OSF.IO/JXE8D and any subsequent arXiv submission constitute the public-posting venue through which sub-task (e) is performed in its rescoped form. The author actively invites domain-expert feedback from researchers in nuclear physics, alpha-cluster theory, computational geometry (especially EDM theory and rigidity theory, which connect to OPEN-SS-37 Route (d)), and discrete mathematics (Steinitz's theorem realizations, Freudenthal–van der Waerden classification literature).

The honesty of this rescope matters: a paper that claims v1.0 status on AI review alone, without making the basis of that status explicit, would mislead readers about the type of validation the paper has received. The conditional-theorem framing throughout the paper (with C5, C6, C7, C8 explicitly registered as paper-level structural hypotheses pending first-principles closure, and OPEN-SS-29, 30, 33, 37 as the corresponding programme-level open problems) is exactly the framing under which the v1.0 designation makes sense: this is a v1.0 *conditional theorem closure paper*, not a v1.0 unconditional derivation.

10 Phase 4 Sketch: Programme-Level Closure Attempts

Once Phase 1 + 3 (the conditional Theorem under v0.1 hypotheses) is solid, Phase 4 attempts to derive C5, C6, and C7 from CPP primitives.

C5 derivation (sketch). C5 says the bound configuration minimizes total energy among physically realizable ones. Minimizing this is straightforward at the level of the formula. The deeper question is: does CPP's lattice Hamiltonian generate this formula structure exactly (each contact face contributes $-B_{\text{pair}}$, additively, with no inter-face couplings beyond what C3 captures)? Likely outcome: Phase 4 reduces "C5 from A1–A11" to "the SS-7 binding formula structure (additivity of B_{pair} contributions across edges) is itself derivable from CPP primitives," which is essentially the formula-derivation question that SS-7 §6.2 already gestures at.

C6 derivation (sketch). C6 says no alpha is interior to the cluster. The derivation route is likely energetic: an interior alpha is "shielded" from external Coulomb but loses some of its B_{pair} contribution channels to other alphas (since fewer faces are exposed for additional contacts in a hypothetical larger cluster); on net, surface configurations are energetically preferred. Working this out from CPP primitives is the OPEN-SS-30 question.

C7 derivation (sketch). Under C6 + cluster contractibility, the cluster's outer 2-surface $\Sigma \cong S^2$ topologically. The alpha-dual embedding maps the contact graph onto Σ. A formal version of this argument (rigorously deriving planarity of the contact graph from the cluster topology) would close OPEN-SS-33 modulo the auxiliary "cluster is contractible" assumption. The contractibility itself follows from the CPP lattice geometry under bound-state assumptions: a non-contractible cluster (e.g., toroidal) would have an internal void at lower density than the surrounding DP-sea, which is energetically unfavorable. Working this out rigorously is the OPEN-SS-33 question.

Common theme. This is consistent with the theme that emerges across the strong-sector programme: the K_3 scale-recurrence (Pattern 6) is the deep structural mystery, and programme-level closure of any specific result tends to reduce to "Pattern 6 holds by construction" or "Pattern 6 is itself the open problem." C5 / C6 / C7 are all candidates for closure-via-Pattern-6-articulation.

11 Physical Interpretation

The simplicial-polytope structure of alpha-cluster nuclei admits a CPP physical interpretation in terms of CPs (Conscious Points), DI-bits (Digital Information bits), and the K_3 collective mode at base-to-base contact. Each alpha is a closed packet of four hybrid-tetrahedral nucleons, each nucleon a CP-cage with definite SSV (Space Stress Vector) orientation and ZBW (Zitterbewegung) phase. At LO, the alpha presents four equilateral outer faces, each face structurally equivalent at LO under the alpha's full T_d symmetry.

When two alphas come into base-to-base contact, the shared face F_{ij} supports a K_3 collective oscillation of the eight CP-cages on the boundary (four from α_i, four from α_j), bonded in a 2-1 polarity arrangement that contributes one nucleon-pair binding quantum $B_{\text{pair}} = M_0/\varphi$ per contact. The contact graph $G(\mathcal{C})$ records which alpha pairs share such a face; the SS-7 binding formula counts edges of G multiplied by B_{pair}, reflecting the additive structure of K_3 collective contributions across non-overlapping contact faces.

Energy minimization at fixed N_α is realized by maximizing the number of K_3 contacts, which is maximizing $|E(G)|$. Under the topological constraint that the cluster forms a closed shell (G planar, S^2 embedding), Euler's formula caps $|E|$ at $3N_\alpha - 6$, with equality iff every face of the planar embedding is a triangle. Steinitz's theorem then identifies G as the 1-skeleton of a convex 3-polytope, and the FvdW classification — itself a deep mathematical fact about uniform-edge-length convex 3-polytopes — identifies the polytope as the unique deltahedron at the eight allowed vertex counts.

The role of refined-C1 facet (b) at $N_\alpha \geq 7$ is the physical mechanism by which an alpha hosts more than four contacts despite presenting only four LO outer faces. Three candidate facet (b) mechanisms remain on the table (face-edge hybrid contact, K_3 delocalization across adjacent faces, partial-overlap docking); each predicts a slightly different geometric realization of degree-5 vertices and is testable against AMD calculations. The SS-7 Table 1 Regime B residual pattern ($\sim +0.55\, B_{\text{pair}}$ at $N_\alpha \in \{7, 8, 9, 10\}$, suppressed to $\sim +0.30\, B_{\text{pair}}$ at $N_\alpha = 12$) is the macroscopic empirical signature of facet (c) (the OPEN-SS-32 cluster-level collective oblate-deformation slip-plane mode), riding on top of the LO geometric structure proved in Theorem 6.1.

11.1 CP/GP Signature at This Scale

Load-bearing axiom identification. The SS-9 conditional theorem inherits its CPP-axiomatic basis from SS-5 and SS-7. Load-bearing axioms in the lemma stack are:

- **A4** (conscious-point cage formation, supplying the rigid 3-simplex structure of each alpha at LO via the SS-5 hybrid-tetrahedral nucleon construction projected to the alpha scale).
- **A5** (propagation efficiency, supplying the $1/\varphi$ factor in $B_{\text{pair}} = M_0/\varphi$).
- **A8′** (cage-volume scaling, supplying $M_0 = m_e z/\varphi$ at the SS-2 / SM-8 inheritance tier).
- **A11** (lattice-scale grounding, fixing MeV units via Λ_{QCD}).

The conditional-theorem hypotheses C5, C6, C7 are not directly identified with single A-axioms; their first-principles derivation from A1–A11 is the OPEN-SS-29 / OPEN-SS-30 / OPEN-SS-33 programme. C1′ facet (b) is registered with multiple-mechanism candidacy at the SS-7 / SS-9 inheritance tier; its first-principles derivation likely shares Layer-3 ancestry with facet (c) (OPEN-SS-32).

Visible-vs-smoothed discreteness. The discrete CPP lattice structure is *visible* in the SS-9 result through:

- The integer edge count $|E| = 3N_\alpha - 6$, a topological invariant of the simplicial-polytope structure — not averaged-out, not smoothed.
- The φ factor in $B_{\text{pair}} = M_0/\varphi$, inherited unchanged from SS-5, traceable through SM-8 to the 600-cell $1/\varphi$ propagation efficiency at the lowest geometric level.
- The $z = 12$ coordination of the 600-cell underlying $M_0 = m_e z/\varphi$.
- The K_3 collective mode itself, a discrete 8-CP coupled oscillation at each contact face.
- The discrete topology of the eight FvdW deltahedra at $N_\alpha \in \{4, 5, 6, 7, 8, 9, 10, 12\}$, with the deltahedra-gap at $N_\alpha \in \{11, 13, 14\}$ as a discrete-structure signature.

The discrete structure is *smoothed* or *averaged* at:

- The single-alpha binding $B_\alpha = 28.296$ MeV, which packages the SS-5 cascade output for the full ^{4}He nucleon arrangement; the underlying discrete CP-cage structure is integrated into a single number at the alpha scale.
- The treatment of alpha rigidity at LO with sub-LO $\sim 5\%$ residual band; the band is a coarse-grained accounting for facet (b) and facet (c) sub-LO corrections that are themselves discrete at the underlying sub-quantum scale.

Thomas's orders-of-magnitude-dilution concern is met here by the discrete topological structure ($3N_\alpha - 6$ edge count, φ factor, $z = 12$ coordination, K_3 recurrence) being preserved across the alpha-scale aggregation: the discreteness is not lost in averaging because the count itself is the prediction.

Macroscopic shadow correspondence. The macroscopic empirical regularity for which the SS-9 simplicial-polytope structure is the sub-quantum CPP shadow is the *alpha-cluster regime of nuclear binding* as observed in the AME 2020 atomic mass evaluation for strict $N{=}Z$ alpha-chain nuclei at $N_\alpha \in \{3, \ldots, 14\}$. Conventional cluster physics (Brink, Ikeda, Wildermuth, Funaki, Freer, Horiuchi; references in SS-7) describes alpha clustering phenomenologically via mean-field-plus-correlation methods or microscopic cluster-model wave functions. What conventional nuclear physics calls "alpha-cluster condensation" or "Hoyle-state-like correlations" is, in CPP, the discrete face-counting $|E| = 3N_\alpha - 6$ produced by the closed-shell topology of base-to-base K_3-bonded alpha-polytopes. The SS-7 zero-parameter agreement to within $\pm 1.5\%$ across twelve nuclei is the macroscopic shadow's quantitative footprint; the SS-9 conditional theorem is the structural derivation of why the shadow has the form it does.

12 CPP-to-Conventional-Physics Mapping

CPP element	Conventional-physics correspondent	Observable signature
Alpha as rigid LO 3-simplex (refined-C1 facet a)	^{4}He cluster as building block in alpha-cluster model (Brink, Ikeda)	Single-alpha binding B_α = 28.296 MeV
Multi-faceted rigidity facet (b) at degree-≥ 5 vertices	Cluster-shape accommodation in microscopic cluster model (AMD, Brink–Bloch)	Contact-distance distribution at degree-5 sites of FvdW deltahedra
K_3 collective mode at contact face (C3)	Nucleon-pair binding contribution per shared face	$B_{\text{pair}} = 2.342$ MeV per edge
Contact graph $G(\mathcal{C})$	Cluster connectivity in microscopic cluster model	Edge count entering total binding
Simplicial-polytope structure (Theorem 6.1)	Closed alpha-polytope topology	$\lvert E\rvert = 3N_\alpha - 6$ (Euler / Steinitz / FvdW)
Cluster-level collective oblate-deformation slip-plane mode (refined-C1 facet c, OPEN-SS-32)	Oblate cluster deformation in cluster physics (Tohsaki–Itagaki 2018)	SS-7 Table 1 Regime B residuals: $\sim +0.55\,B_{\text{pair}}$ at $N_\alpha \in \{7, 8, 9, 10\}$, $\sim +0.30\,B_{\text{pair}}$ at $N_\alpha = 12$
Deltahedra-gap at $N_\alpha \in \{11, 13, 14\}$ (OPEN-SS-31)	Cluster shape transitions in heavy alpha-conjugate nuclei	-0.26% to -0.73% residuals at ^{44}Ti, ^{52}Fe, ^{56}Ni vs. LO formula

Table 1: Structural correspondence between SS-9 CPP-mechanistic elements and conventional nuclear-physics descriptions. The mapping is structural, not literal: CPP operates at finer granularity (CP-cages, DI-bit propagation) than the mean-field or microscopic-cluster descriptions in the conventional column. The correspondences are at the level of symmetry, degrees of freedom, and observable predictions.

13 Conclusion

SS-9 v0.1 ships the first formal LaTeX conditional theorem closing C4 (the simplicial-polytope contact structure assumption used implicitly in SS-7) on the refined-C1 (multi-faceted alpha rigidity, SS-7 v1.3) foundation. The theorem holds at $N_\alpha \in \{4, 5, 6, 7, 8, 9, 10, 12\}$, conditional on three new paper-level hypotheses C5 (energy minimization), C6 (cluster surface-realization), and C7 (contact-graph planarity), each registered as a candidate open problem for first-principles closure from CPP axioms (OPEN-SS-29, OPEN-SS-30, OPEN-SS-33). The proof routes through three lemmas (Lemma 3.1, Lemma 4.1, Lemma 5.1) and the Freudenthal–van der Waerden classification of convex deltahedra. Refined-C1 facet (b) is load-bearing in the geometric-realization clause at $N_\alpha \geq 7$ where the FvdW deltahedron has degree-≥ 5 vertices.

The deltahedra-gap range $N_\alpha \in \{11, 13, 14\}$, where clauses (i)–(iii) of the Theorem hold but clause (iv) does not, is registered as OPEN-SS-31. The $N_\alpha = 3$ planar degenerate case is handled separately by direct edge count. Refined-C1 facet (c) (cluster-level collective oblate-deformation slip-plane mode, OPEN-SS-32) is treated as an NLO addition to the LO binding formula and does not affect the LO geometric proof structure; the parallel OPEN-SS-32 / U-shape investigation thread (Sessions 13–23, Phases 1–11) is summarized at §8.3, with Phase 8 Refinement A as standing best refinement at v0.1 ship and single-session R3-channel refinement candidates declared exhausted.

13.1 Swarm-Validation Contribution

Predictions added. SS-9 is a derivation paper, not a prediction paper: its primary contribution is a conditional theorem advancing OPEN-SS-24, not a new set of zero-parameter empirical predictions. The derivation enables structural promotion of the twelve SS-7 zero-parameter alpha-chain predictions from "edge-count assumed empirically" to "edge-count derived conditional on C5 + C6 + C7 + C8 + C1′ + C2 + C3." SS-9 thus reinforces the SS-7 swarm contribution by demoting C4 from a free hypothesis to a derived conclusion (under the named conditionals), with the conditionals themselves now subject to first-principles closure attempts.

Running swarm total. The CPP programme's cumulative count of zero-parameter empirical correspondences as of SS-9 v0.1 ship is dominated by the SS-7 12 alpha-chain predictions (RMS 0.80%), the SS-5 light-nuclei predictions, the SM-8 quark-mass predictions, the SM-3 SU(3) results, and the SS-2 nucleon-structure observables. SS-9 does not add new zero-parameter predictions but *strengthens the structural foundation* of the SS-7 12-prediction subset by providing a conditional derivation of the simplicial structure they depend on.

Implausibility-of-accident statement. The probability that the SS-7 12-prediction agreement at zero parameters with RMS 0.80% arises from accident scales as (residual band/typical parameter space)^N at the swarm size $N = 12$ (SS-7 alpha-chain alone), and is astronomically small; SS-9 v0.1 makes this implausibility-of-accident reasoning more robust by reducing the remaining structural-hypothesis count from one (C4 free) to three (C5, C6, C7 free, all candidates for first-principles closure). Each subsequent closure of OPEN-SS-29 / OPEN-SS-30 / OPEN-SS-33 will further strengthen the SS-7 swarm contribution.

13.2 Problem Status After This Paper

- OPEN-SS-24 (derivation of simplicial contact structure from CPP): **ADVANCED to conditional theorem**. C4 promoted from B-tier structural hypothesis (SS-7 v1.3 status) to conditional theorem at C5 + C6 + C7 + C8 + C1′ + C2 + C3 inheritance tier (SS-9 v0.5 status). Unconditional promotion pending closure of OPEN-SS-29 / OPEN-SS-30 / OPEN-SS-33 / OPEN-SS-37.

- OPEN-SS-29 (C5 first-principles derivation): **REGISTERED**. Programme-level closure target.
- OPEN-SS-30 (C6 first-principles derivation): **REGISTERED**. Programme-level closure target.
- OPEN-SS-31 (deltahedra-gap structural realization at $N_\alpha \in \{11, 13, 14\}$): **REGISTERED** (carried over from earlier sketches; first formal registration here). Programme-level closure target.
- OPEN-SS-32 (refined-C1 facet (c) attenuation factor derivation, cluster-level slip-plane mode): **ACTIVE INVESTIGATION**, separate thread. Standing best refinement: Phase 8 Refinement A (factor 3.6 polytope-residual improvement, 48% of empirical scale captured); single-session R3-channel refinement candidates EXHAUSTED at Session 23 close (Phases 9 + 10 ruled out σ-parameterized K_3 class; Phase 11 R3-Pauli structurally redundant). Forward path: sub-shell-physics decomposition (multi-paper, candidate SS-10 on shell-corrected baselines) and alternate-channel work (R4-DP-sea, SR-tensor, finite-A SEMF corrections).
- OPEN-SS-33 (C7 first-principles derivation): **REGISTERED (NEW this paper)**. Programme-level closure target.
- OPEN-ORG-012 (.tex conversion of SS-9 v0.3 working draft): **RETIRED** at SS-9 v0.1 ship.

Acknowledgements

The SS-9 paper realizes OPEN-SS-24, registered originally in SS-7 v1.0 (19 April 2026) as the natural follow-on derivation of the simplicial-polytope contact structure. Thomas Lee Abshier ND directed the paper's role in the strong-sector programme and authorized the OPEN-ORG-012 promotion at Session 23 Phase 11 close (6 May 2026), recognizing that Phase 11 NULL marked the natural saturation point for single-session R3-channel work and that locking SS-9 v0.1 first creates a stable reference for the forthcoming SS-10 sub-shell-physics paper.

Development arc 16 April 2026 – 6 May 2026. The v0.2 sketch (Session 2, 16 April 2026) introduced the Steinitz-theorem identification and a centroid-hull supporting-hyperplane construction for Lemma B. Session 3 surfaced an internal-consistency concern with the strict-C1 reading at degree-5 vertices, which led to the multi-faceted rigidity refinement integrated into SS-7 v1.3 (26 April 2026). The v0.3 working draft (1 May 2026, Session 16 finalisation) replaced the v0.2 supporting-hyperplane construction with the C7 + Steinitz routing, dissolving the v0.2 supporting-hyperplane gap and registering OPEN-SS-33. v0.1 .tex (this paper) preserves the v0.3 mathematical content and adds: the formal LaTeX presentation, the §8.3 OPEN-SS-32 / U-shape investigation status report integrating Sessions 13–23 Phases 1–11, the §11.1 CP/GP signature subsection (per PD-001, 24 April 2026), the §13.1 swarm-validation contribution (per PD-001), and the §12 CPP-to-conventional-physics mapping table.

OPEN-SS-32 / U-shape investigation thread. The eleven phases of the OPEN-SS-32 / U-shape investigation thread (Sessions 13–23, 1–6 May 2026) ran in parallel with SS-9 development. Each phase produced primary patches plus the standard six-step §15 documentation chain (substantive, Step A + Step C session log + Vignette, Step B + Step D transcript + reasoning, Step E Research Frontier + future projects, Step H paste-ready handover). The thread's positive scoping outcomes (Phase 5 channel pass, Phase 6 5% smooth-A bullseye, Phase 8

Refinement A factor 3.6 polytope-residual improvement with near-exact ^{40}Ca and ^{36}Ar matches), seven sequential ruling-outs producing 12 programme-level negative results, and Phase 11 structural-redundancy null result are summarized in §8.3. The methodological category "structural-redundancy null result" was introduced at Phase 11 close as the third F1-pass / F3-* pattern (rejection-at-point, rejection-across-family, structural-redundancy).

External reviewer team. SS-9 v0.1 ships before formal external review. The intended round-1 reviewer panel (ChatGPT, Copilot, with rotation as appropriate) will engage v0.1 against the standard CPP review-protocol expectations (paper-formatting.md §§4.1A and 4.1B explicit; full Lemma stack scrutinized; conditional-theorem framing validated); reviewer responses will inform v0.2 / v0.3 revisions. The symmetric-honesty protocol (operating_system.md §4 Phase 4, adopted 19 April 2026 from the SS-7 review cycle) will govern reviewer engagement.

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